O.S. PART II
516.
DIF

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

PART II FRAME FIELDS

DIFFERENTIAL GEOMETRY

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

DIFFERENTIAL GEOMETRY PART II

FRAME FIELDS

O.U. COURSE UNITS COLLECTION
FOR REFERENCE ONLY

A commentary on Chapter II of O'Neill's
Elementary Differential Geometry
Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

Course Team

Chairman:	Mr. P.E.D. Strain	Lecturer in Mathematics
	Dr. R.A. Bailey	Course Assistant in Mathematics
With assistance from:		
	Dr. J.M. Aldous	Senior Lecturer in Mathematics
	Mr. G.J. Burt	Lecturer in Educational Technology
	Dr. P.M. Clark	Lecturer in Physics
	Mr. P.B. Cox	Student Computing Service
	Dr. F.C. Holroyd	Lecturer in Mathematics
	Mr. T.C. Lister	Staff Tutor in Mathematics
	Mr. R.J. Margolis	Staff Tutor in Mathematics
	Dr. C.A. Rowley	Course Assistant in Mathematics
	Mr. M.G. Simpson	Course Assistant in Mathematics
Consultants:	Prof. S. Robertson	Professor of Pure Mathematics, University of Southampton
	Prof. T. Willmore	Professor of Pure Mathematics, University of Durham

BAR CODE No.
+ 1
CLASS No.
516.36 DIF
17 SEP 19[illegible]
179752
LOAN CAT NL

The Open University Press, Walton Hall, Milton Keynes

First published 1975

Copyright © 1975 The Open University

All rights reserved. No part of this work may be reproduced in any form, by mimeograph or any other means without permission in writing from the publishers.

Produced in Great Britain by
The Open University Press

ISBN 335 05700 4

This text forms part of the correspondence element of an Open University Third Level Course. The complete list of parts in the course is given at the end of this text.

For general availability of supporting material referred to in this text, please write to the Director of Marketing, The Open University, P.O. Box 81, Milton Keynes, MK7 6AT.

Further information on Open University courses may be obtained from The Admissions Office, The Open University, P.O. Box 48, Milton Keynes, MK7 6AB.

1.1

CONTENTS

		Page
	Set Book	4
	Bibliography	4
	Conventions	4
II.1	Introduction and Dot Product	5
II.2	Curves	16
II.3	The Frenet Formulas	28
II.4	Arbitrary-Speed Curves	38
II.5	Covariant Derivatives	49
II.6	Frame Fields	55
II.7	Connection Forms	59
II.8	The Structural Equations	61
	Partial Differentiation	66
	Further Exercises and Solutions	72

Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition, (W.A. Benjamin/Addison-Wesley 1973).

R.C. Smith and P. Smith: *Mechanics,* SI edition (John Wiley, 1972).

Also mentioned in this Part is:
T. Wilmore: *An Introduction To Differential Geometry* (O.U.P., 1964).

Conventions

Before starting work on this text, please read M334 Part Zero. Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory by* E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

References to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

II.1 INTRODUCTION AND DOT PRODUCT

Introduction

This whole section depends only on Sections I.1 and I.2, and may be read easily even if you are not entirely confident about the later sections of Chapter I.

It provides a revision of the basic vector concepts that you have met in MST 282 and M201. You should make sure that you are familiar with the following ideas: the *dot product, cross product, triple scalar product* and *norm* for E^3 (*Unit MST 282 1, Some Basic Tools,* Section 2); the *dot product, norm, Schwarz inequality, angle, orthogonality, orthonormal bases* and *orthonormal expansions* for general Euclidean vector spaces (*Unit M201 16, Euclidean Spaces I: Inner Products*); *orthogonal matrices* (*Unit M201 24, Orthogonal and Symmetric Transformations,* Section 1.2). You will also find it useful to revise the technique of evaluation of a 3 × 3 determinant (KKOP, page 684).

These vector ideas are revised and then applied to the tangent vector spaces $T_p(E^3)$. For example, the dot product of two tangent vectors with the same point of application is defined by

$$v_p \cdot w_p = v \cdot w.$$

We find that the usual vector space results carry over to each of the tangent spaces. Later on we shall extend these definitions to vector fields by the pointwise principle: for example, if V and W are vector fields the dot product of V and W, V·W, will be the function from E^3 to R whose value at p is V(p)·W(p).

READ: Introduction to Chapter II and Section II.1 (pages 42–48).

Comment

(i) *Page 47: the determinant and cross product* The determinant is a function from the set of all square real matrices into the real numbers. It is written by replacing the brackets of the matrix by straight lines. The rules for evaluating it on 2 × 2 and 3 × 3 matrices are as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{21}\, a_{12};$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{aligned} & a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}\,a_{23} \\ & - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23} \end{aligned}$$

(see KKOP, page 684)

which may be written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

This last expression gives the expansion of the formal determinant for v × w as

$$U_1(p)\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} + U_2(p)\begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix} + U_3(p)\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= (v_2w_3 - v_3w_2)\,U_1(p) + (v_3w_1 - v_1w_3)\,U_2(p) + (v_1w_2 - v_2w_1)\,U_3(p)$$

$$= (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)_p.$$

O'Neill assumes that you are familiar with the following properties of determinants:

(a) a determinant is linear as a function of each of its rows;

(b) interchanging two rows of a determinant multiplies its values by − 1;

(c) a determinant has value zero if and only if its rows are linearly dependent vectors.

In *Unit M 201 5, Determinants and Eigenvalues,* you learnt these properties for the *columns* of a determinant. However, in Section 1.5 of that unit you saw that the determinant of the transpose of a matrix is the same as the determinant of the original matrix, and thus all these results are true for rows as well.

By Lemma II.1.8, v × w is orthogonal to both v and w, and

$$\|v \times w\|^2 = (v\cdot v)\,(w\cdot w) - (v\cdot w)^2$$
$$= \|v\|^2\,\|w\|^2 - (v\cdot w)^2.$$

The angle ϑ between v and w is defined by

$$v\cdot w = \|v\|\|w\| \cos\vartheta$$

(see page 44), and so

$$\|v \times w\|^2 = \|v\|^2\|w\|^2 - (\|v\|\,\|w\| \cos\vartheta)^2$$
$$= \|v\|^2\|w\|^2\,(1 - \cos^2\vartheta)$$
$$= \|v\|^2\|w\|^2 \sin^2\vartheta.$$

Taking square roots,

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| (\sin^2 \vartheta)^{\frac{1}{2}}.$$

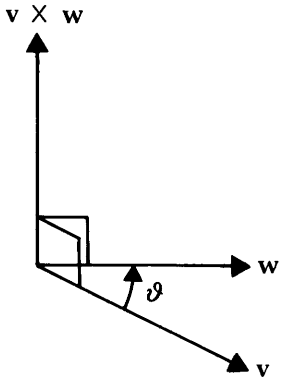
For any given value of $\cos \vartheta$ (in $[-1,1]$) we can choose ϑ to lie in the range $0 \leqslant \vartheta \leqslant \pi$, that is ϑ is the smaller angle between v and w; for ϑ in this range $\sin \vartheta$ is non-negative and so $(\sin^2 \vartheta)^{\frac{1}{2}} = \sin \vartheta$. Thus

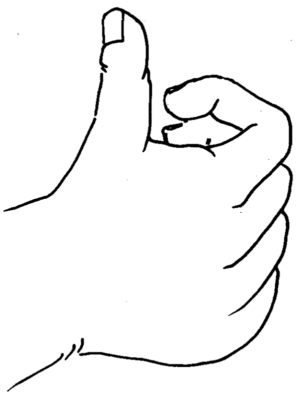
$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin \vartheta$$

and so

$$\mathbf{v} \times \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \sin \vartheta \, \mathbf{n}$$

where **n** is a unit vector orthogonal to both **v** and **w**.





In fact n is obtained from v and w by the right-hand rule (page 48, penultimate paragraph), and so O'Neill's definition of cross-product is equivalent to that given in *Unit MST 282 1, Some Basic Tools.*

One consequence of the definition of cross-product is the following.

Lemma II.1.A If e_1 and e_2 are orthogonal unit tangent vectors and $e_3 = e_1 \times e_2$ then e_3 is a unit tangent vector orthogonal to both e_1 and e_2 and

$$e_1 \times e_2 = e_3; \qquad e_2 \times e_1 = -e_3;$$

$$e_2 \times e_3 = e_1; \qquad e_3 \times e_2 = -e_1;$$

$$e_3 \times e_1 = e_2; \qquad e_1 \times e_3 = -e_2.$$

In particular, these formulas hold when $e_1 = U_1(p)$, $e_2 = U_2(p)$, in which case $e_3 = U_1(p) \times U_2(p) = U_3(p)$.

There is no reason to restrict the definition of cross product to tangent vectors. As for the dot product, we can define it for pairs of points of E^3 also. If we put $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0)$, $\mathbf{u}_3 = (0, 0, 1)$ then for points $\mathbf{p}$ and $\mathbf{q}$ of E^3 we define the cross product of $\mathbf{p}$ and $\mathbf{q}$ to be

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

$$= (p_2q_3 - p_3q_2)\mathbf{u}_1 + (p_3q_1 - p_1q_3)\mathbf{u}_2 + (p_1q_2 - p_2q_1)\mathbf{u}_3$$

$$= (p_2q_3 - p_3q_2, p_3q_1 - p_1q_3, p_1q_2 - p_2q_1).$$

Useful results about determinants and cross products are contained in the following exercises, which you should attempt now.

1. Page 49, Exercise 4.
2. Page 49, Exercise 5 (first part).
3. Page 49, Exercise 6.

Additional Text

The result of the following exercise will be needed later.

4. Page 49, Exercise 7.

Summary

Notation

$\mathbf{p} \cdot \mathbf{q}$	Page 42, Definition 1.1
$\mathbf{v}_\mathbf{p} \cdot \mathbf{w}_\mathbf{p}$	Page 44, Definition 1.3
$\|\mathbf{p}\|$	Page 43, line 9
$\|\mathbf{v}_\mathbf{p}\|$	Page 44, line 12
$d(\mathbf{p}, \mathbf{q})$	Page 43, Definition 1.2
$\mathcal{N}_\epsilon$	Page 43, line -4
${}^tA$	Page 47, line 1
$\mathbf{v}_\mathbf{p} \times \mathbf{w}_\mathbf{p}$	Page 47, Definition 1.7
$\mathbf{p} \times \mathbf{q}$	Text, page 8

Definitions

(i)	Dot product of points $\mathbf{p} \cdot \mathbf{q}$ of tangent vectors $\mathbf{v} \cdot \mathbf{w}$	Page 42, Definition 1.1 Page 44, Definition 1.3
(ii)	Norm of a point $\|\mathbf{p}\|$ of a tangent vector $\|\mathbf{v}\|$	Page 43, line 9 Page 44, line 12
(iii)	Euclidean distance $d(\mathbf{p}, \mathbf{q})$	Page 44, Definition 1.2

(iv)	ϵ-neighbourhood $\mathcal{N}_\epsilon$	Page 43, line -4
(v)	Open set	Page 43, line -3
(vi)	Orthogonal vectors	Page 44, line -7
(vii)	Unit vector	Page 44, line -6
(viii)	Frame, or orthonormal basis	Page 44, Definition 1.4
(ix)	Orthonormal expansion	Page 45, line -1
(x)	Attitude matrix	Page 46, Definition 1.6
(xi)	Orthogonal matrix	Page 46, line -2
(xii)	Transpose of A, ${}^tA$	Page 47, line 1
(xiii)	Cross product of points $p \times q$ of tangent vectors $v \times w$	Text, page 8 Page 47, Definition 1.7
(xiv)	Triple scalar product $u \cdot v \times w$	Page 48, line -3 of text

Results

(i)	$\|p + q\| \leqslant \|p\| + \|q\|$ (the triangle inequality) and $\|ap\| = \|a\|\,\|p\|$.	Page 43, line 11
(ii)	$\|v \cdot w\| \leqslant \|v\|\,\|w\|$ (Schwarz inequality).	Page 44, line 13
(iii)	$U_1(p)$, $U_2(p)$, $U_3(p)$ constitute a frame at p.	Page 45, line 6
(iv)	Orthonormal expansion. If e_1, e_2, e_3 is a frame at p and $v \in T_p(E^3)$, then $v = (v\cdot e_1)\,e_1 + (v\cdot e_2)\,e_2 + (v\cdot e_3)\,e_3.$	Page 45, Theorem 1.5
(v)	If e_1, e_2, e_3 is a frame, $v = \Sigma\, a_i e_i$ and $w = \Sigma\, b_i e_i$, then $v\cdot w = a_1 b_1 + a_2 b_2 + a_3 b_3.$	Page 46, line 7
(vi)	If A is an orthogonal matrix, ${}^tA = A^{-1}$.	Page 47, line 5
(vii)	The dot product is symmetric, bilinear, positive definite.	Page 43, lines 1-7
(viii)	The cross product is alternating, bilinear.	Page 47, line -11
(ix)	$\|v \times w\| = (v\cdot v\; w\cdot w - (v\cdot w)^2)^{\frac{1}{2}} = \|v\|\,\|w\| \sin \vartheta.$	Page 47, Lemma I.8 and Page 48, line -9
(x)	$v \times w$ is orthogonal to v and to w.	Page 47, Lemma I.8
(xi)	$v \times w = \|v\|\,\|w\| \sin \vartheta\, n$, where **n** is obtained by the right-hand rule.	Text, page 7
(xii)	$v \times w \neq 0$ if and only if v and w are linearly independent.	Page 49, Exercise 5
(xiii)	$u\cdot v \times w \neq 0$ if and only if **u**, v and w are linearly independent.	Page 49, Exercise 4(b)
(xiv)	Interchanging any two of **u**, **v**, **w** reverses the sign of $u\cdot v \times w$ but does not change its absolute value.	Page 49, Exercise 4(c)
(xv)	If e_1, e_2, e_3 is a frame, $e_1 \cdot e_2 \times e_3 = \pm 1.$	Page 49, Exercise 6

(xvi) If $\mathbf{e}_1, \mathbf{e}_2$ are orthogonal unit vectors and $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, then $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a frame, and

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3;\ \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3;$$
$$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1;\ \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1;$$
$$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2;\ \mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2.$$

Text, page 7

(xvii) If $\mathbf{u}$ is a unit vector, any vector $\mathbf{v}$ may be expressed uniquely as

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u})\,\mathbf{u} + \mathbf{v}_2,$$

where $\mathbf{v}_2$ is orthogonal to $\mathbf{u}$.

Page 49, Exercise 7

Techniques

(i)	Evaluation of dot and cross products and norms.	Page 42, Definition 1.1 and Page 47, Definition 1.7
(ii)	Evaluation of triple scalar products.	Page 49, Exercise 4(a)
(iii)	Expansion in terms of an orthonormal basis.	Page 45, Theorem 1.5
(iv)	Determination of the attitude matrix and use of the result ${}^tA = A^{-1}$.	Page 46, Definition 1.6

Exercises

Technique (i)

5. Page 48, Exercise 1.

Technique (ii)

6. If $\mathbf{v}$ and $\mathbf{w}$ are as in the preceding exercise and $\mathbf{u} = (-3, 1, -1)$ at the same point, compute $\mathbf{u}\cdot\mathbf{v} \times \mathbf{w}$.

Technique (iii)

7. Page 49, Exercise 3.

Technique (iv)

8. Find the attitude matrix A of the frame in the preceding exercise. What is its inverse?

Solutions

1. Page 49, Exercise 4.

(a) Suppose **u**, **v**, **w** are tangent vectors at the point **p**.

$\mathbf{u} = u_1U_1(p) + u_2U_2(p) + u_3U_3(p)$.

$\mathbf{v} \times \mathbf{w} = c_1U_1(p) + c_2U_2(p) + c_3U_3(p) \qquad (*)$

where the c_i are found from

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$\mathbf{u}\cdot\mathbf{v} \times \mathbf{w} = c_1u_1 + c_2u_2 + c_3u_3$

which is (*) with $U_i(p)$ replaced by u_i for $i = 1, 2, 3$: thus replacing $U_i(p)$ by u_i in the determinant gives $\mathbf{u}\cdot\mathbf{v} \times \mathbf{w}$.

(b) $\mathbf{u}\cdot\mathbf{v} \times \mathbf{w} \neq 0 \iff$ the rows of the determinant are linearly independent

$\iff$ **u**, **v** and **w** are linearly independent.

(c) Again, this follows from the properties of determinants: if any two rows in a determinant are interchanged the whole determinant changes sign.

(d) $\mathbf{u} \times \mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{u} \times \mathbf{v}$, because the dot product is symmetric,

$= \mathbf{u}\cdot\mathbf{v} \times \mathbf{w}$, by part (c).

2. Page 49, Exercise 5.

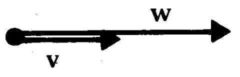
We know that $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\,\|\mathbf{w}\| \sin\vartheta$, where ϑ is the angle between **v** and **w**.

$\mathbf{v} \times \mathbf{w} = \mathbf{0} \iff \|\mathbf{v} \times \mathbf{w}\| = 0$

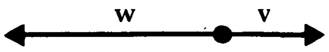
$\iff \|\mathbf{v}\| = 0 \quad \text{or} \quad \|\mathbf{w}\| = 0 \quad \text{or} \quad \sin\vartheta = 0$

$\iff \mathbf{v} = \mathbf{0} \quad \text{or} \quad \mathbf{w} = \mathbf{0} \quad \text{or} \quad \vartheta$ is a multiple of π.

If $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ then **v** and **w** are certainly linearly dependent. If ϑ is a multiple of π then



$\vartheta = 0$



$\vartheta = \pi$

v and **w** are collinear, and so linearly dependent. Conversely, if **v** and **w** are linearly dependent, either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ or $\sin\vartheta = 0$.

Thus $\mathbf{v} \times \mathbf{w} \neq \mathbf{0} \iff$ **v** and **w** are linearly independent.

3. Page 49, Exercise 6.

$e_2 \times e_3$ is orthogonal to both e_2 and e_3. e_2 and e_3 span a plane in $T_p(E^3)$, so the tangent vectors orthogonal to both e_2 and e_3 form a line in $T_p(E^3)$. We know that e_1 is orthogonal to both e_2 and e_3 and so e_1 spans this line: thus $e_2 \times e_3$ must be a multiple of e_1.

$$\|e_2 \times e_3\| = \|e_2\|\,\|e_3\| \sin\frac{\pi}{2} = 1,$$

so

$$e_2 \times e_3 = \pm e_1.$$

Thus

$$e_1 \cdot e_2 \times e_3 = \pm e_1 \cdot e_1 = \pm 1.$$

If A is an orthogonal matrix, then it is the attitude matrix of a frame e_1, e_2, e_3, where the coordinates of e_i with respect to $U_1(p)$, $U_2(p)$, $U_3(p)$ form the ith row of A for i = 1, 2, 3. By the first part of the question, $e_1 \cdot e_2 \times e_3 = \pm 1$. However, by Exercise 4(a) on page 49,

$$e_1 \cdot e_2 \times e_3 = \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix} = \det A$$

so the determinant of A is ± 1.

We could have reached this last result in another way, using the facts (see *Unit M201 5, Determinants and Eigenvalues,* Section 1) that if A and B are square matrices of the same size then

$$\det({}^tA) = \det(A)$$

and

$$\det(AB) = \det(A)\det(B).$$

Now, if A is orthogonal, ${}^tAA = I$

so

$$\det({}^tAA) = \det(I) = 1$$

so

$$\det({}^tA)\det(A) = 1$$

so

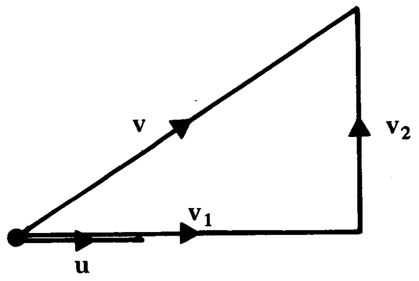
$$\det(A)\det(A) = 1$$

so

$$\det(A) = \pm 1.$$

4. Page 49, Exercise 7.

Choose $v_1 = (v \cdot u)u$, $v_2 = v - v_1$. Then certainly $v = v_1 + v_2$ and v_1 is the component of v in the u direction, by definition. Moreover, $v - v_1$ is the only possible choice for v_2 in order that $v = v_1 + v_2$, and so this expression is certainly unique.



It remains to prove just that $v_1 \cdot v_2 = 0$. Now

$$\begin{aligned} v_1 \cdot v_2 &= v_1 \cdot (v - v_1) \\ &= (v \cdot u)u \cdot (v - (v \cdot u)u) \\ &= (v \cdot u)(u \cdot v - (v \cdot u)u \cdot u) \\ &= (v \cdot u)(u \cdot v - v \cdot u), \qquad \text{because } u \cdot u = 1, \\ &= 0, \qquad \text{because } u \cdot v = v \cdot u. \end{aligned}$$

5. Page 48, Exercise 1.

(a) $v \cdot w = 1.(-1) + 2.0 + (-1).3 = -1 - 3 = -4.$

(b)

$$v \times w = \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ 1 & 2 & -1 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= (2.3 - (-1).0, (-1).(-1) - 1.3, 1.0 - 2.(-1))$$

$$= (6, -2, 2).$$

(c) $\|v\| = (1^2 + 2^2 + (-1)^2)^{\frac{1}{2}} = \sqrt{6}$; $\frac{v}{\|v\|} = \frac{(1, 2, -1)}{\sqrt{6}}$.

$\|w\| = ((-1)^2 + 0^2 + 3^2)^{\frac{1}{2}} = \sqrt{10}$; $\frac{w}{\|w\|} = \frac{(-1, 0, 3)}{\sqrt{10}}$.

(d) $\|v \times w\| = (6^2 + (-2)^2 + 2^2) = \sqrt{44} = 2\sqrt{11}.$

Alternatively, we can use Lemma II.1.8:

$$\|v \times w\|^2 = (v \cdot v)(w \cdot w) - (v \cdot w)^2 = \|v\|^2\|w\|^2 - (v \cdot w)^2 = 6.10 - (-4)^2 = 44.$$

(e) If ϑ is the angle between v and w,

$$v \cdot w = \|v\| \, \|w\| \cos\vartheta$$

so $$-4 = \sqrt{6}\sqrt{10} \cos\vartheta$$

so $$\cos\vartheta = \frac{-4}{\sqrt{60}} = \frac{-4}{2\sqrt{15}} = \frac{-2}{\sqrt{15}}.$$

6.

$$u \cdot v \times w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} -3 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= -3(2.3 - (-1).0) + 1((-1)^2 - 1.3) - 1(1.0 - 2.(-1))$$

$$= -3.6 + 1.(-2) - 1.2 = -22.$$

Alternatively, since v × w has already been calculated in Exercise 5, we could go straight to

$$\begin{aligned} u \cdot v \times w &= (-3, 1, -1) \cdot (6, -2, 2) \\ &= -3.6 + 1.(-2) + (-1).2 = -22. \end{aligned}$$

7. Page 49, Exercise 3.

$$e_1 \cdot e_1 = \frac{1}{6}(1^2 + 2^2 + 1^2) = 1$$

$$e_2 \cdot e_2 = \frac{1}{8}((-2)^2 + 0^2 + 2^2) = 1$$

$$e_3 \cdot e_3 = \frac{1}{3}(1^2 + (-1)^2 + 1^2) = 1$$

$$e_1 \cdot e_2 = \frac{1}{\sqrt{6}}\frac{1}{\sqrt{8}}(-2 + 0 + 2) = 0$$

$$e_2 \cdot e_3 = \frac{1}{\sqrt{8}}\frac{1}{\sqrt{3}}(-2 + 0 + 2) = 0$$

$$e_3 \cdot e_1 = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{6}}(1 - 2 + 1) = 0$$

This shows that e_1, e_2, e_3 constitutes a frame.

By Theorem II.1.5,

$$v = (v \cdot e_1)e_1 + (v \cdot e_2)e_2 + (v \cdot e_3)e_3.$$

$$v \cdot e_1 = \frac{1}{\sqrt{6}}(6 + 2 - 1) = \frac{7}{\sqrt{6}};$$

$$v \cdot e_2 = \frac{1}{\sqrt{8}}(-12 - 2) = \frac{-14}{\sqrt{8}};$$

$$v \cdot e_3 = \frac{1}{\sqrt{3}}(6 - 1 - 1) = \frac{4}{\sqrt{3}};$$

$$v = \frac{7}{\sqrt{6}}e_1 - \frac{14}{\sqrt{8}}e_2 + \frac{4}{\sqrt{3}}e_3.$$

We can check this by working out the right-hand side directly: we obtain

$$\frac{7}{\sqrt{6}}\frac{(1, 2, 1)}{\sqrt{6}} - \frac{14}{\sqrt{8}}\frac{(-2, 0, 2)}{\sqrt{8}} + \frac{4}{\sqrt{3}}\frac{(1, -1, 1)}{\sqrt{3}}$$

$$= \frac{7}{6}(1, 2, 1) - \frac{14}{8}(-2, 0, 2) + \frac{4}{3}(1, -1, 1)$$

$$= \left(\frac{7}{6}, \frac{7}{3}, \frac{7}{6}\right) + \left(\frac{7}{2}, 0, -\frac{7}{2}\right) + \left(\frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right)$$

$$= (6, 1, -1) = v.$$

8. We simply take the coordinates of e_1, e_2, e_3 and write them as the rows of A. Thus

$$A = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{8}} & 0 & \frac{2}{\sqrt{8}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Since A is an attitude matrix it is orthogonal, and so we know ${}^tA = A^{-1}$: that is, A^{-1} may be obtained from A by interchanging corresponding rows and columns.

This gives

$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{8}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{8}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

II.2 CURVES

Introduction

This section depends heavily on Sections I.4 and II.1 but not at all on the later sections of Chapter I.

Here we take up the story of curves left in Section I.4, and extend the definitions and results of Section II.1 to vector fields defined on curves. We immediately introduce a way of differentiating these vector fields; in particular this gives us the *acceleration* of a curve: the velocity and acceleration turn out to be the same as those of a particle travelling along the curve, as given in *Unit MST 282 2, Kinematics,* Section I.1.

All of this is simply giving us enough technical apparatus to establish the Frenet formulas, which are a core feature of the course, in the next section.

READ: Section II.2 (pages 51–55).

Comments

(i) Pages 51–52: unit-speed reparametrization This particular type of reparametrization is very important. From now on, whenever O'Neill mentions a "unit-speed reparametrization" he means not just a reparametrization which happens to have unit speed but a reparametrization of the type given in the proof of Theorem 2.1. There is a useful symmetry in this situation. If β is a unit-speed reparametrization of α we have

$$\beta(s) = \alpha(t(s));$$

in this particular case s and t are inverse functions, and so we also have

$$\beta(s(t)) = \alpha(t):$$

that is, α is also a reparametrization of β. We shall often make use of this fact.

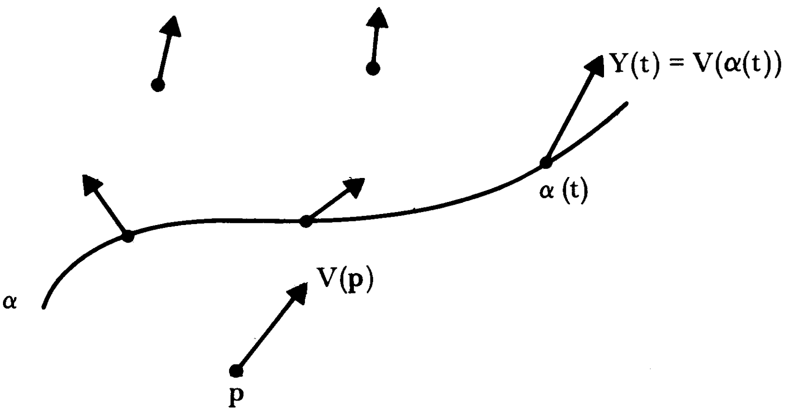
Theorem 2.1 is important because it is often sufficient to know that a unit-speed reparametrization of a curve α exists, without computing it explicitly. It is this theorem which allows many proofs to begin: "We may assume α has unit speed". This considerably simplifies many things, but you should always check that the remark is justified.

(ii) Page 52: Definition 2.2 A vector field on a curve should be distinguished from a vector field on E^3. It is true that if α is a curve and V is a vector field on E^3 then the function Y defined by

$$Y(t) = V(\alpha(t))$$

is a vector field on α (see the diagram opposite); however not every vector field on α is necessarily the restriction of a vector field on E^3. For example, it may happen that α crosses itself, that is there are different values t_1 and t_2 such that

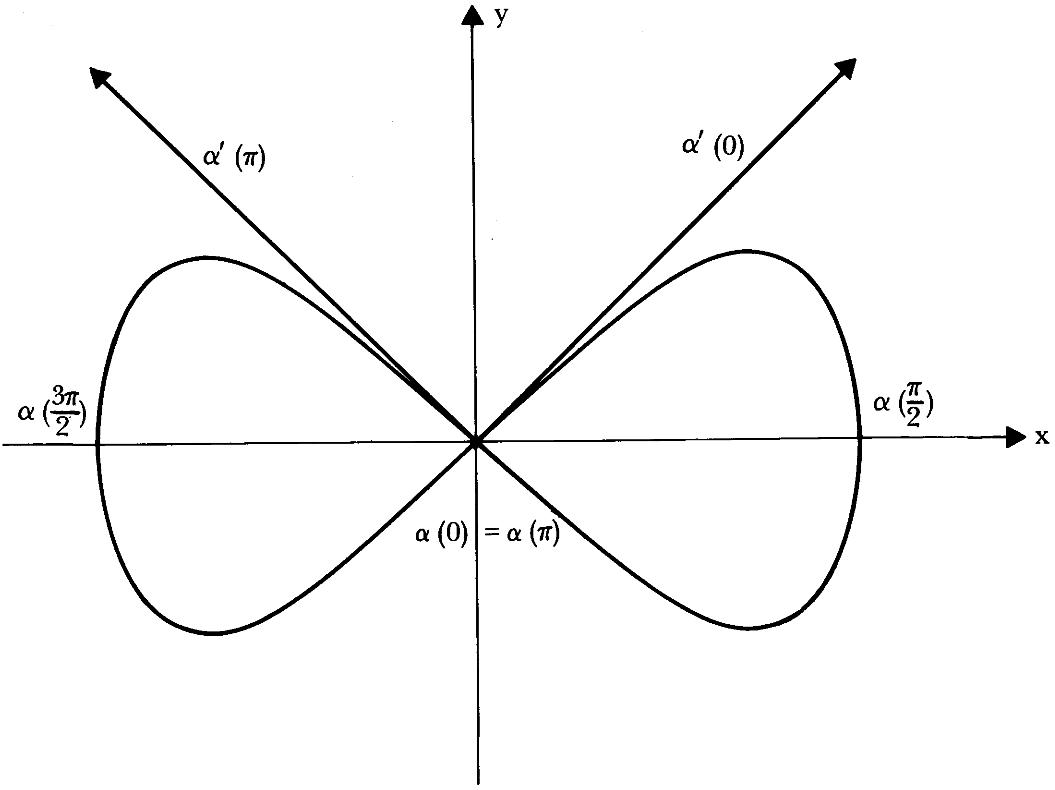
$$\alpha(t_1) = \alpha(t_2) = p;$$



if Y is a vector field on α it is not obliged to take the same values at t_1 and t_2, but if V is a vector field on E^3 then it has a unique value at p and so $V(\alpha(t_1)) = V(\alpha(t_2)) = V(p)$. A good example of this is provided by the figure-of-eight curve given by

$$\alpha(t) = (\sin t, \tfrac{1}{2}\sin 2t,\ 0) \qquad (t \in R)$$

which crosses itself at $0 = \alpha(0) = \alpha(\pi)$: at this point the vector field α' on α does take different values, for $\alpha'(0) = (1, 1, 0)$ while $\alpha'(\pi) = (-1, 1, 0)$.



(iii) Page 54: Lemma 2.3 It is important at this point to be quite clear about O'Neill's slightly unusual use of the words "constant", "straight line" and "parallel". A curve α is a *constant* if it is of the form

$$\alpha\colon t \longmapsto c$$

that is, the image of α in E^3 is a single point. A curve α is *straight line* if and only if it is of the form

$$\alpha\colon t \longmapsto p + tq$$

for some points p and q with $q \neq 0$. The image of α is certainly the line joining p and p + q. Intuitively we might think of the curves

$$\beta\colon t \longmapsto p + 2tq$$

and

$$\gamma: t \longmapsto p + t^3 q$$

as the same as α, because their images are also the straight line joining **p** and **p** + **q**, but it is important to realize that, under Definition I.4.1, they are different curves from α. In fact β is the reparametrization of α by the function

$$h: t \longmapsto 2t$$

and γ is the reparametrization of α by the function

$$k: t \longmapsto t^3.$$

Are β and γ straight lines? Well β is, because β can be written

$$\beta: t \longmapsto p + t(2q),$$

but γ cannot be written in the form

$$\gamma: t \longmapsto a + tb$$

for any points **a** and **b** so γ is not, technically speaking, a straight line, merely a reparametrization of one.

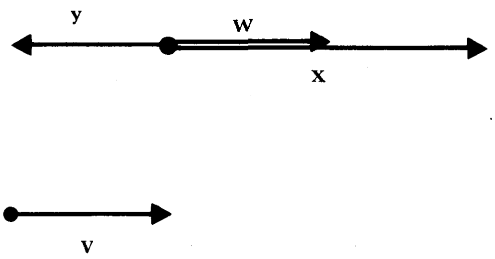
Finally, "parallel" has the meaning given on page 6, that of having the same vector parts. This is a more restrictive definition than that usually used in elementary geometry. For example, if

$$v = (1, 0, 0)_{(0, 0, 0)}$$

$$w = (1, 0, 0)_{(1, 1, 0)}$$

$$x = (2, 0, 0)_{(1, 1, 0)}$$

$$y = (-1, 0, 0)_{(1, 1, 0)}$$



then **v** is parallel to **w** but not **x** or **y**, even though they all point along the same direction. For tangent vectors at the same point we say they are *collinear* if they point along the same direction, that is if one is a scalar multiple of the other. The vectors **w**, **x** and **y** are all collinear.

(iv) Curves – a Convention From now on we adopt the convention used by Spivak that a function defined on a subset of E^n has the whole of E^n as its domain unless otherwise specified. Thus in Exercise 1 on page 55 the curve

$$\alpha(t) = (2t, t^2, \frac{t^3}{3})$$

is taken to be the curve

$$\alpha: t \longmapsto (2t, t^2, \frac{t^3}{3}) \qquad (t \in R).$$

Additional Text

The following exercises introduce results which will be needed later.

Attempt them now.

1. Page 56, Exercise 7.
2. Y and Z are vector fields on a curve α. Prove that
   $(Y \times Z)' = Y' \times Z + Y \times Z'$.

Supplementary Comment

(i) Page 51: the "standard theorem of calculus" This is a combination of several results from Spivak, which you should be able to accept.

Summary

Notation

v	Page 51, line 4
s	Page 51, line −7
Y	Page 52, Definition 2.2
α''	Page 54, line 7

Definitions

(i)	Speed v	Page 51, line 4
(ii)	Arc length	Page 51, line 11
(iii)	Arc-length function s	Page 51, line −7
(iv)	Arc-length parametrization	Page 52, line 7
(v)	Unit-speed reparametrization	Text, page 16
(vi)	Orientation-preserving reparametrization	Page 52, line −12
(vii)	Orientation-reversing reparametrization	Page 52, line −11
(viii)	Vector field (Y) on a curve	Page 52, Definition 2.2
(ix)	Euclidean coordinate functions of a vector field on a curve	Page 53, line 9
(x)	Derivative of a vector field on a curve	Page 54, line 3
(xi)	Acceleration (α'') of a curve	Page 54, line 7
(xii)	Parallel vector field	Page 54, line −10
(xiii)	Collinear tangent vectors	Text, page 18

Results

(i)	Any regular curve has a unit-speed reparametrization.	Page 51, Theorem 2.1
(ii)	If β is a unit-speed reparametrization of α, then α is a reparametrization of β.	Text, page 16
(iii)	If α has unit speed, then $s(t) = t$.	
(iv)	The linearity and Leibnizian properties of differentiation of vector fields on curves. $(aY + bZ)' = aY' + bZ'$ $(fY)' = f'Y + fY'$ $(Y\cdot Z)' = Y'\cdot Z + Y\cdot Z'$ $(Y \times Z)' = Y' \times Z + Y \times Z'$.	Page 54, lines 12-17 Text, page 19
(v)	α is constant if and only if $\alpha' = 0$.	Page 54, Lemma 2.3(1)
(vi)	α is a straight line if and only if $\alpha' \neq 0$ and $\alpha'' = 0$.	Page 55, Lemma 2.3(2)
(vii)	$\lVert Y \rVert$ is constant if and only if $Y\cdot Y' = 0$.	Page 54, lines -14 to -12
(viii)	Y is parallel if and only if $Y' = 0$.	Page 55, Lemma 2.3(3)
(ix)	$Y(h)' = h'Y'(h)$.	Page 56, Exercise 7

Techniques

(i)	Calculation of arc-length.	Page 51, line 11
(ii)	Finding an arc-length function $s = s(t)$.	Page 51, proof of Theorem 2.1
(iii)	Unit-speed reparametrization.	Page 51, Theorem 2.1
(iv)	Expression of a vector field on a curve in terms of its Euclidean coordinate functions.	Page 53, line 7
(v)	Algebra of vector fields on a curve.	Page 53, last paragraph
(vi)	Differentiation of vector fields on a curve	
	(a) from first principles	Page 54, line 3
	(b) using linearity and Leibnizian properties.	Page 54, lines 12-17 and Text, page 19
(vii)	Calculation of velocity, speed, acceleration.	Page 51, line 4 and Page 54, line 7

Exercises

Technique (i)

3. Page 55, Exercise 4. (Note that "log" means "natural logarithm".)

Techniques (ii) and (iii)

4. Page 55, Exercise 3.

Techniques (iv) and (v)

5. Page 55, Exercise 6.

Techniques (vi)(a) and (vii)

6. Page 55, Exercise 2. The required cone is the set of points in E^3 satisfying $x^2 + y^2 = z^2$.

Technique (vi)(b)

7. Let Y and Z be the vector fields on the curve $\alpha(t) = (\cos t, \sin t, t)$ given by

   $Y(t) = (\cos t, \sin t, 1)_{\alpha(t)}$,

   $Z(t) = (\sin t, t, \cos t)_{\alpha(t)}$.

   Find $Y'(t)$ and $Z'(t)$ and hence $V'(t)$ when V is each of the following:

   (a) $V(t) = Y(t) + Z(t)$
   (b) $V(t) = t^2 Y(t)$
   (c) $V = Y \cdot Z$
   (d) $V = Y \times Z$.

Theory Exercises (omit if you are short of time).

8. Y is a vector field on a curve α. Prove that if $\|Y\|$ is constant and Y' is never zero then Y'' is never orthogonal to Y. (HINT: Differentiate $Y \cdot Y$ = constant.)

9. Page 56, Exercise 9. (HINT: For the "if" part of (b), let γ be a unit-speed reparametrization of α and show that γ is a straight line by showing that γ'' is both collinear and orthogonal to γ'. By result (ii) α is a reparametrization of γ.)

Solutions

1. Page 56, Exercise 7.

   Y(t) is a tangent vector at $\alpha(t)$, so $Y(h(s))$ is a tangent vector at $\alpha(h(s))$, so $Y(h)$ is a vector field on the curve $\alpha(h)$.

   We can write $Y(t) = (y_1(t), y_2(t), y_3(t))_{\alpha(t)}$

   so

$$Y'(t) = (y_1'(t), y_2'(t), y_3'(t))_{\alpha(t)}.$$

Now

$$Y(h)(s) = (y_1(h(s)), y_2(h(s)), y_3(h(s)))_{\alpha(h(s))}$$

so

$$Y(h)'(s) = (y_1'(h(s))\,h'(s), y_2'(h(s))h'(s), y_3'(h(s))h'(s))_{\alpha(h(s))}$$

by the definition of differentiation of a vector field on a curve and by the chain rule

$$= h'(s)(y_1'(h(s)), y_2'(h(s)), y_3'(h(s)))_{\alpha(h(s))}$$

$$= h'(s)\,Y'(h(s))$$

so

$$Y(h)' = h'Y'(h).$$

Note that this result is, not surprisingly, yet another form of the chain rule!

2. If

$$Y(t) = (y_1(t), y_2(t), y_3(t))$$

and

$$Z(t) = (z_1(t), z_2(t), z_3(t))$$

then

$$(Y \times Z)(t) = (y_2(t)z_3(t) - y_3(t)z_2(t), y_3(t)z_1(t) - y_1(t)z_3(t), y_1(t)z_2(t) - y_2(t)z_1(t)).$$

To differentiate $Y \times Z$ we differentiate its coordinate functions. The first of these is

$$y_2z_3 - y_3z_2,$$

whose derivative is

$$y_2'z_3 + y_2z_3' - y_3'z_2 - y_3z_2',$$ by the Leibnizian rule for derivatives of functions from R to R,

$$= (y_2'z_3 - y_3'z_2) + (y_2z_3' - y_3z_2').$$

The other two coordinate functions are similar, and so we obtain

$$\begin{aligned}(Y \times Z)'(t) &= (y_2'(t)z_3(t) - y_3'(t)z_2(t), y_3'(t)z_1(t) - y_1'(t)z_3(t), y_1'(t)z_2(t) - y_2'(t)z_1(t))\\ &\quad + (y_2(t)z_3'(t) - y_3(t)z_2'(t), y_3(t)z_1'(t) - y_1(t)z_3'(t), y_1(t)z_2'(t) - y_2(t)z_1'(t))\\ &= Y'(t) \times Z(t) + Y(t) \times Z'(t)\\ &= (Y' \times Z + Y \times Z')(t).\end{aligned}$$

Thus

$$(Y \times Z)' = Y' \times Z + Y \times Z'.$$

3. Page 55, Exercise 4.

$$\alpha(t) = (2t, t^2, \log t),\ t > 0.$$

$$\alpha(1) = (2, 1, 0) = p$$

$$\alpha(2) = (4, 4, \log 2) = q,$$

so the curve passes through p and q.

$$\alpha'(t) = \left(2, 2t, \frac{1}{t}\right)_{\alpha(t)}$$

$$\|\alpha'(t)\| = \left(4 + 4t^2 + \frac{1}{t^2}\right)^{\frac{1}{2}} = 2t + \frac{1}{t} \text{ since } t > 0.$$

The arc-length between p and q is the arc-length from $t = 1$ to $t = 2$, which is

$$\int_1^2 \left(2t + \frac{1}{t}\right) dt = \Big[t^2 + \log t\Big]_1^2$$

$$= 4 + \log 2 - 1$$

$$= 3 + \log 2.$$

4. Page 55, Exercise 3.

$$\alpha(t) = (\cosh t, \sinh t, t)$$

$$\alpha'(t) = (\sinh t, \cosh t, 1)_{\alpha(t)}$$

$$\|\alpha'(t)\| = (\sinh^2 t + \cosh^2 t + 1)^{\frac{1}{2}} = (2\cosh^2 t)^{\frac{1}{2}}, \qquad \text{because } 1 + \sinh^2 t = \cosh^2 t,$$

$$= \sqrt{2}\cosh t.$$

$$s(t) = \int_0^t \|\alpha'(u)\|\, du = \int_0^t \sqrt{2}\cosh u\, du = \sqrt{2}\sinh t$$

so

$$t(s) = \sinh^{-1}\frac{s}{\sqrt{2}}.$$

The unit-speed reparametrization of α based at $t = 0$ is

$$\beta(s) = \alpha(t(s)) = \left(\cosh\left(\sinh^{-1}\frac{s}{\sqrt{2}}\right), \sinh\left(\sinh^{-1}\frac{s}{\sqrt{2}}\right), \sinh^{-1}\frac{s}{\sqrt{2}}\right)$$

$$= \left(\left(1 + \frac{s^2}{2}\right)^{\frac{1}{2}}, \frac{s}{\sqrt{2}}, \sinh^{-1}\frac{s}{\sqrt{2}}\right),$$

(using the identity $\cosh^2\theta = 1 + \sinh^2\theta$ for $\theta = \sinh^{-1}\frac{s}{\sqrt{2}}$),

5. Page 55, Exercise 6.

(a) $$\begin{aligned} Y(t) &= 0 - \alpha(t) \quad \text{at } \alpha(t) \\ &= (-\cos t, -\sin t, -t)_{\alpha(t)} \\ &= -\cos t\ U_1(\alpha(t)) - \sin t\ U_2(\alpha(t)) - t\ U_3(\alpha(t)) \\ &= -\cos t\ U_1 - \sin t\ U_2 - t\ U_3 \end{aligned}$$

using the convention mentioned in the middle of page 53.

(b) $$\alpha'(t) = (-\sin t, \cos t, 1)_{\alpha(t)}$$

$$\alpha''(t) = (-\cos t, -\sin t, 0)_{\alpha(t)}$$

$$Y(t) = (\cos t - \sin t)U_1 + (\cos t + \sin t)U_2 + U_3.$$

(c) Any tangent vector at $\alpha(t)$ orthogonal to both $\alpha'(t)$ and $\alpha''(t)$ is a scalar multiple of $\alpha'(t) \times \alpha''(t)$, if this is not zero.

$$\alpha'(t) \times \alpha''(t) = (\sin t, -\cos t, 1)_{\alpha(t)}$$

$$\|\alpha'(t) \times \alpha''(t)\|^2 = \sin^2 t + \cos^2 t + 1 = 2$$

so a unit vector in the direction of $\alpha'(t) \times \alpha''(t)$ is $\pm\ (1/\sqrt{2})\alpha'(t) \times \alpha''(t)$.

$$\text{Thus } Y(t) = \pm\left(\frac{\sin t\ U_1}{\sqrt{2}} - \frac{\cos t\ U_2}{\sqrt{2}} + \frac{U_3}{\sqrt{2}}\right).$$

(d) $$\begin{aligned} Y(t) &= (\alpha(t+\pi) - \alpha(t))_{\alpha(t)} \\ &= ((\cos(t+\pi), \sin(t+\pi), t+\pi) - (\cos t, \sin t, t))_{\alpha(t)} \\ &= ((-\cos t, -\sin t, t+\pi) - (\cos t, \sin t, t))_{\alpha(t)} \\ &= -2\cos t\ U_1 - 2\sin t\ U_2 + \pi U_3. \end{aligned}$$

6. Page 55, Exercise 2.

Let $p = \alpha(t) = (t \cos t, t \sin t, t)$.

The usual reason for the presence of the sine and cosine functions in the Euclidean coordinate functions of a curve is to take advantage of the identity $\sin^2\theta + \cos^2\theta = 1$. Thus we see that

$$p_1^2 + p_2^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = p_3^2,$$

that is, every point p on the curve satisfies

$$p_1^2 + p_2^2 = p_3^2. \qquad (*)$$

If C is the set of points satisfying (*), C intersects the plane $p_3 = r$ in the circle $p_1^2 + p_2^2 = r^2$, for each real number r. Thus C is the circular cone with axis the z-axis, vertex the origin, and half-angle $\frac{\pi}{4}$. Every point of the curve lies on C; the vertex (0, 0, 0) is $\alpha(0)$.

$$\alpha'(t) = (\cos t - t \sin t, \sin t + t \cos t, 1)_{\alpha(t)}$$

$$\alpha'(0) = (1, 0, 1)_{(0, 0, 0)}$$

$$\|\alpha'(0)\| = (1^2 + 1^2)^{\frac{1}{2}} = \sqrt{2};$$

$$\alpha''(t) = (-2\sin t - t \cos t, 2\cos t - t \sin t, 0)_{\alpha(t)}$$

$$\alpha''(0) = (0, 2, 0)_{(0, 0, 0)}.$$

7. We first evaluate Y' and Z', differentiating each coordinate function

$$Y(t) = (\cos t, \sin t, 1)_{\alpha(t)}$$

$$Y'(t) = (-\sin t, \cos t, 0)_{\alpha(t)}$$

$$Z(t) = (\sin t, t, \cos t)_{\alpha(t)}$$

$$Z'(t) = (\cos t, 1, -\sin t)_{\alpha(t)}.$$

(a) $V(t) = Y(t) + Z(t)$

so $V'(t) = Y'(t) + Z'(t)$, by linearity,

$$= (\cos t - \sin t, \cos t + 1, -\sin t)_{\alpha(t)}.$$

(b) $V(t) = t^2\, Y(t)$

so

$V'(t) = 2t\, Y(t) + t^2 Y'(t)$, by the Leibnizian property,

$$= 2t\,(\cos t, \sin t, 1)_{\alpha(t)} + t^2(-\sin t, \cos t, 0)_{\alpha(t)}$$

$$= (2t \cos t - t^2 \sin t, 2t \sin t + t^2 \cos t, 2t)_{\alpha(t)}.$$

(c) $V(t) = Y(t)\cdot Z(t)$

so

$V'(t) = Y'(t)\cdot Z(t) + Y(t)\cdot Z'(t)$, by the Leibnizian property,

$$= (-\sin t, \cos t, 0)\cdot(\sin t, t, \cos t) + (\cos t, \sin t, 1)\cdot(\cos t, 1, -\sin t)$$

$$= -\sin^2 t + t \cos t + 0 + \cos^2 t + \sin t - \sin t$$

$$= \cos^2 t - \sin^2 t + t \cos t.$$

(d) $V(t) = Y(t) \times Z(t)$

so

$$V'(t) = Y'(t) \times Z(t) + Y(t) \times Z'(t), \qquad \text{by the Leibnizian property,}$$

$$= (-\sin t, \cos t, 0)_{\alpha(t)} \times (\sin t, t, \cos t)_{\alpha(t)}$$

$$+ (\cos t, \sin t, 1)_{\alpha(t)} \times (\cos t, 1, -\sin t)_{\alpha(t)}$$

$$= (\cos^2 t, \sin t \cos t, -t \sin t - \sin t \cos t)_{\alpha(t)} +$$

$$(-1 - \sin^2 t, \cos t + \sin t \cos t, \cos t - \sin t \cos t)_{\alpha(t)}$$

$$= (\cos^2 t - \sin^2 t - 1, \cos t + 2\sin t \cos t, \cos t - t \sin t - 2\sin t \cos t)_{\alpha(t)}$$

$$= (-2\sin^2 t, \cos t + \sin 2t, \cos t - t\sin t - \sin 2t)_{\alpha(t)}.$$

8. If $\|Y\|$ has the constant value c, then

$$Y \cdot Y = c^2.$$

Differentiation gives

$$Y' \cdot Y + Y \cdot Y' = 0$$

so

$$Y \cdot Y' = 0.$$

Further differentiation gives

$$Y' \cdot Y' + Y \cdot Y'' = 0.$$

If $Y(t)$ is orthogonal to $Y''(t)$ then $(Y \cdot Y'')(t) = 0$. This forces $(Y' \cdot Y')(t) = 0$, that is $\|Y'(t)\| = 0$. The norm of a vector is zero if and only if that vector is itself zero, so this implies $Y'(t) = \mathbf{0}$, which is contrary to assumption. Thus $Y(t)$ cannot be orthogonal to $Y''(t)$.

9. Page 56, Exercise 9.

(a) $\|\alpha'\|$ is a constant

$\iff$ $\alpha' \cdot \alpha'$ is a constant

$\iff$ $(\alpha' \cdot \alpha')' = 0$

$\iff$ $\alpha'' \cdot \alpha' + \alpha' \cdot \alpha'' = 0$

$\iff$ $\alpha' \cdot \alpha'' = 0$

$\iff$ α'' is always orthogonal to α'.

(b) Suppose β is the straight line $t \mapsto p + tq$, and α is the reparametrization $\beta(h)$ of β. Then

$$\alpha(u) = p + h(u)q$$

$$\alpha'(u) = h'(u)q_{\alpha(u)}$$

$$\alpha''(u) = h''(u)q_{\alpha(u)}.$$

Then $\alpha'(u)$ and $\alpha''(u)$ are both scalar multiples of the tangent vector $q_{\alpha(u)}$, and so they are collinear. Conversely, we wish to show that the collinearity of α' and α'' implies that α is a reparametrization of a straight line. By the remark about symmetry in Comment (i), it is enough to show that α has a unit-speed reparametrization $\gamma = \alpha(h)$ which is a straight line. Consequently we choose γ to be a unit-speed reparametrization of α: Theorem II.2.1 tells us that we can choose such a γ.

Now $\gamma(s) = \alpha(h(s))$

$\gamma'(s) = h'(s)\,\alpha'(h(s))$, by Lemma I.4.5,

$\gamma''(s) = h''(s)\,\alpha'(h(s)) + (h'(s))^2\,\alpha''(h(s))$, by the Leibnizian property.

$\alpha''(h(s))$ and $\alpha'(h(s))$ collinear

$\Rightarrow$ $\alpha''(h(s))$ is a scalar multiple of $\alpha'(h(s))$

$\Rightarrow$ $\gamma''(s)$ is a scalar multiple of $\alpha'(h(s))$

$\Rightarrow$ $\gamma''(s)$ and $\gamma'(s)$ are collinear.

However, γ has unit speed, that is $\|\gamma'\| = 1$, so part (a) tells us that $\gamma''(s)$ is orthogonal to $\gamma'(s)$. The only way that $\gamma''(s)$ can be both collinear and orthogonal to $\gamma'(s)$ is that $\gamma''(s) = 0$: hence by Lemma II.2.3(2) γ is a straight line.

II.3 THE FRENET FORMULAS

Introduction

This section carries straight on from Section II.2. The Frenet formulas for unit-speed curves are the first really important result in the course. You should try to become thoroughly familiar with the concepts of *tangent, principal normal, binormal, curvature* κ and *torsion* τ, which are collectively called the Frenet apparatus, and be able to find them for any unit-speed curve.

We use these ideas to define the Frenet approximation to a curve, which is a type of Taylor approximation. You have met Taylor approximations before in *Unit M100 14, Sequences and Limits II* or *Unit MST 281 10, Taylor Approximation,* and *Unit M231 13, Taylor Approximations.* The Frenet approximation is less important than the Frenet apparatus, except in so far as it helps one to visualize the shape of a curve near a point and suggests various characterizations of curves in terms of their curvature and torsion. Such a *characterization* is a statement of the form:

> β is a curve of a particular type (e.g. plane curve, cylindrical helix) if and only if the curvature (κ) and torsion (τ) of β satisfy a particular condition (e.g. $\tau = 0$; or τ/κ is constant).

The fact that there are such characterizations indicates why κ and τ are important functions associated with a curve: we shall see just how important they are in Chapter III. The section concludes with some characterizations of curves: you should note the methods of proof as much as the results. Further characterizations follow in the exercises and in the next section.

READ: Section II.3 (pages 56–63).

Comments

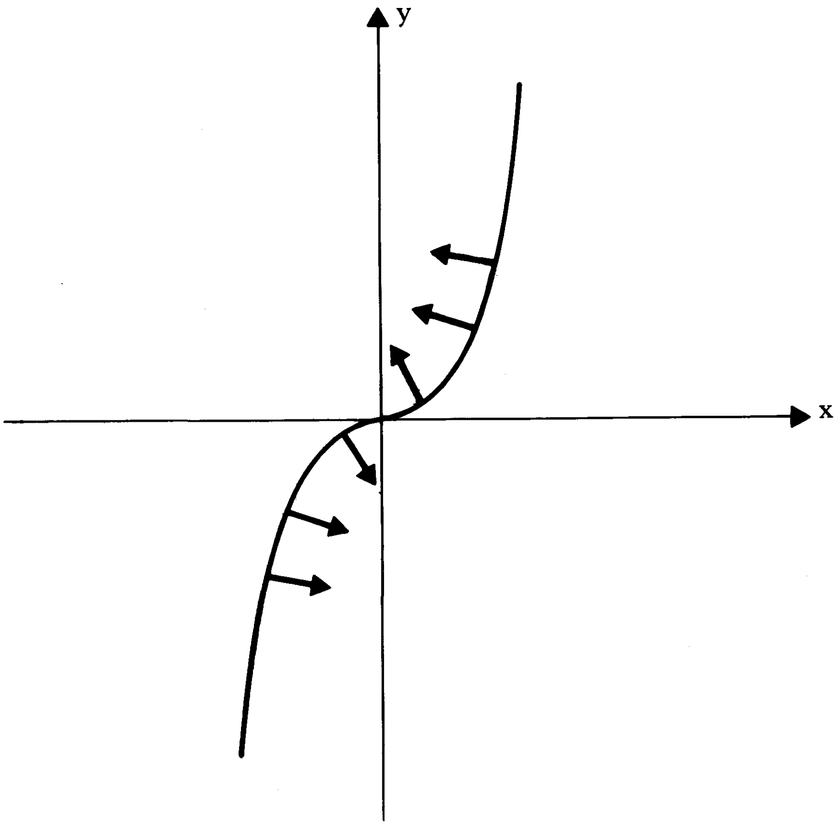
(i) Pages 57–8: curvature and principal normal The definition of κ as $\|T'\|$ forces us to avoid points where κ is zero, for at such a point not only is N undefined but κ may not be differentiable (because the square-root function is not differentiable at zero). Moreover, N may suddenly change from one side of the curve to the other near a point where κ is zero, as shown in the diagram opposite.

Consequently the footnote on page 57 is very important: we restrict our attention to portions of curves where the curvature is positive.

It is possible to develop a theory where κ is allowed to become zero and even negative, in which N is defined for many curves even at points where $\kappa = 0$. In this theory κ and N are still differentiable at such points. O'Neill does not take this approach (except for curves in E^2 – see the optional section in the exercises), but if you are interested you can read about it in T. J. Willmore: *An Introduction to Differential Geometry* (O.U.P. 1964), Chapter I.

(ii) Page 57: binormal We choose B to be T × N specifically to obtain a vector field which is orthogonal to T and N at each point. Since T(s) and N(s) are orthogonal unit vectors, Lemma II.1.A tells us that B(s) is a unit vector orthogonal to T(s) and N(s), and moreover the following equations hold:

$$\begin{aligned} T \times N &= B; & N \times T &= -B; \\ N \times B &= T; & B \times N &= -T; \\ B \times T &= N; & T \times B &= -N. \end{aligned}$$



Since T(s), N(s), B(s) form a frame at each point $\beta(s)$ we can extend the idea of orthonormal expansion by the pointwise principle to obtain the result that any vector field Y on β has an orthonormal expansion of the form

$$Y = (Y \cdot T)T + (Y \cdot N)N + (Y \cdot B)B.$$

(iii) Page 59 onwards: the correspondence between points and tangent vectors

The identification of the tangent vector v_p with the point v, for working which is concerned just with the Euclidean coordinates, enables us to omit the subscripts to tangent vectors. This convention considerably simplifies manipulation and proofs. For example, O'Neill uses this convention throughout the proof of Corollary 3.5: without this convention the last line of page 61 would read

$$\beta'(s) \cdot q_{\beta(s)} = \beta''(s) \cdot q_{\beta(s)} = 0$$

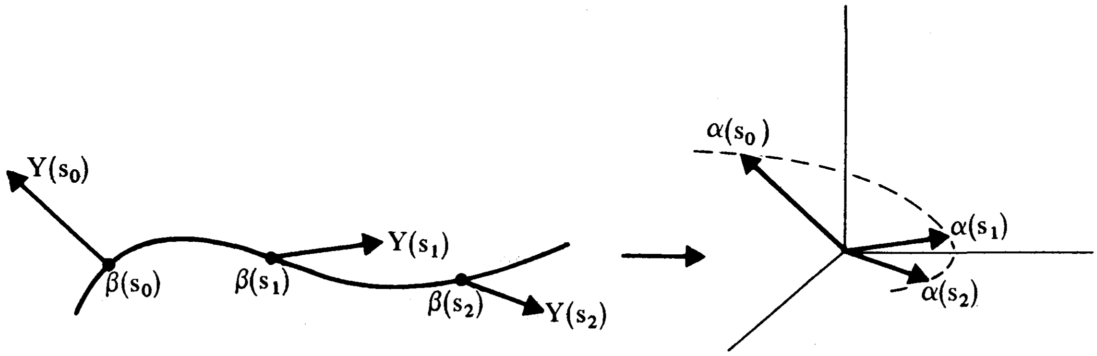
and the second paragraph on page 62 would contain words to the effect that since B is parallel there is a fixed vector v such that $B(s) = v_{\beta(s)}$ and would then define f(s) as $(\beta(s) - \beta(0)) \cdot v$; this adds verbiage and symbols to the proof without enhancing our understanding of it. Consequently we use this convention whenever it is possible to do so without causing confusion. We note that by page 58 O'Neill is already writing

$$T(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)$$

rather than the technically more correct

$$T(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)_{\left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ \frac{bs}{c}\right)}.$$

In the last paragraph on page 59 we take this identification of tangent vectors with points a little further. We associate the tangent vector $Y(s) = (y_1(s), y_2(s), y_3(s))_{\beta(s)}$ with the point $(y_1(s), y_2(s), y_3(s))$ of E^3.



We simply take all the tangent vectors Y(s) and move the point of application $\beta(s)$ to the origin; the new end points of the tangent vectors become the new curve. If we call this new curve α, then α is parametrized in such a way that $\alpha(s_0)$ is the point with the same Euclidean coordinates as the tangent vector $Y(s_0)$. This technique allows us to transfer certain results about curves to vectors fields on curves, and vice versa.

Additional Text

The following exercise introduces a result that we shall need in the next section, so attempt the exercise now. Do not spend too long on it, and be sure to read the solution.

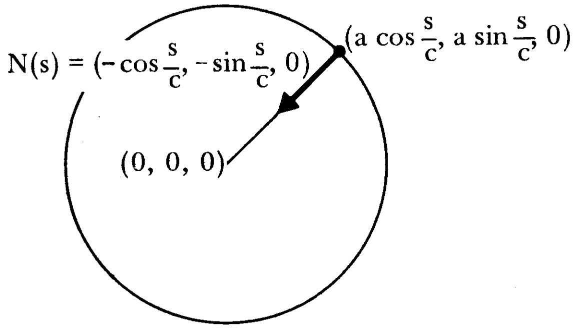
1. Page 65, Exercise 7.

Supplementary Comments

(i) Page 57: line -2 The "orthonormal expansion" referred to is that of B′ in terms of T, N, B, as mentioned in Comment (ii).

$$B' = (B'\cdot T)T + (B'\cdot N)N + (B'\cdot B)B$$
$$= (B'\cdot N)N \quad \text{if } B'\cdot T = B'\cdot B = 0.$$

(ii) Page 59: line 5 Looked at from above we have:


$N(s) = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0)$
$(a\cos\frac{s}{c}, a\sin\frac{s}{c}, 0)$
(0, 0, 0)

The normal is in the opposite direction to the projection of the position vector $\beta(s)$ on the xy-plane, and so it always points towards the z-axis.

LANCASHIRE POLYTECHNIC LLRS

(iii) Page 61: line 4 This is not the Taylor approximation to β of degree 3: that would be

$$\beta(s) \sim \beta(0) + (s - \kappa_0^2 s^3)T_0 + (\kappa_0 \frac{s^2}{2} + \frac{d\kappa}{ds}(0)s^3)N_0 + \kappa_0 \tau_0 \frac{s^3}{6} B_0 .$$

We are interested in the approximation of the component of $\beta(s) - \beta(0)$ in each of the directions T_0, N_0, B_0, so we keep just the term of lowest degree in each component.

Summary

Notation

T	Page 56, line -4
κ	Page 57, line 2
N	Page 57, line 6
B	Page 57, line 8
τ	Page 58, line 3

Definitions

(i)	Unit tangent T	Page 56, line -4
(ii)	Curvature κ	Page 57, line 2
(iii)	Principal normal N	Page 57, line 6
(iv)	Binormal B	Page 57, line 8
(v)	Frenet frame field T, N, B	Page 57, Lemma 3.1
(vi)	Torsion τ	Page 58, line 3
(vii)	Frenet approximation	Page 61, line 5
(viii)	Tangent line	Page 61, line 10
(ix)	Osculating plane	Page 61, line 14
(x)	Frenet apparatus T, N, B, κ, τ	
(xi)	Plane curve	Page 61, line -12
(xii)	Spherical curve	Page 63, last two lines of text

Example

(i) The unit-speed helix

$$\beta(s) = \left(a \cos\frac{s}{c}, a \sin\frac{s}{c}, \frac{bs}{c}\right),$$

where $c^2 = a^2 + b^2$, has constant curvature a/c^2 and constant torsion b/c^2. Page 58, Example 3.3

Results

(i) The Frenet formulas.

$$\begin{aligned} T' &= \quad\ \ \kappa N \\ N' &= -\kappa T \qquad\quad + \tau B \\ B' &= \qquad\quad -\tau N \end{aligned}$$

Page 58, Theorem 3.2

(ii) The Frenet approximation: for small s,

$$\beta(s) \sim \beta(0) + sT(0) + \kappa(0)\frac{s^2}{2} N(0) + \kappa(0)\tau(0)\frac{s^3}{6} B(0).$$

Page 61, line 4

(iii) A unit-speed curve with positive curvature is a plane curve if and only if $\tau = 0$. Page 61, Corollary 3.5

(iv)	A unit-speed curve has zero torsion and constant curvature κ if and only if it is part of a circle with radius $1/\kappa$.	Page 59, third paragraph, and Page 62, Lemma 3.6
(v)	A spherical unit-speed curve has curvature $\geqslant 1/a$, where a is the radius of the sphere.	Page 63, last paragraph
(vi)	If $\bar{\alpha} = \alpha(h)$ and α are both unit-speed curves, then $\bar{T} = \varepsilon T(h)$, $\bar{N} = N(h)$, $\bar{B} = \varepsilon B(h)$, $\bar{\kappa} = \kappa(h)$, $\bar{\tau} = \tau(h)$, where $\varepsilon = h' = \pm 1$.	Page 65, Exercise 7
(vii)	$T = N \times B$, $N = B \times T$, $B = T \times N$.	Text, page 28
(viii)	If Y is a vector field on α, $Y = (Y \cdot T)T + (Y \cdot N)N + (Y \cdot B)B$.	Text, page 29

Techniques

(i)	Finding the Frenet apparatus T, N, B, κ, τ, by differentiation, followed by taking norms and cross products.	Page 58, Example 3.3
(ii)	Identification of a vector field on a curve with another curve.	Page 59, last paragraph
(iii)	The following two techniques are useful in characterizations:	
(a)	to show a function is constant, show that its derived function is zero, simplifying the derived function by using the Frenet formulas;	Page 62, lines 6-11
(b)	prove that "type of curve" ⇒ "properties of κ and τ" first, as the proof should give you information which will indicate the method of proving the converse.	Page 62, proof of Lemma II.3.6

Exercises

Technique (i)

2. Page 63, Exercise 1. It is sufficient to show that the curve is part of a circle, and find its centre and radius.

Technique (ii)

3. For β as in the preceding exercise, let σ be the curve whose coordinate functions are given by $\sigma(s) = T(s)$. Find the speed of σ.

Technique (iii)

4. Page 65, Exercise 10.

Optional Section – Curves in the Plane

5. Curves in the plane are treated somewhat differently from those in E^3. If you would like to make a small sub-study of them, read the definitions given in Exercise 8 and then do that exercise. More exercises on curves in the plane will be provided later. Note that, because E^2 is two-dimensional, N spans the set of vectors orthogonal to T: T′ is orthogonal to T because $\|T\|$ = constant, and so $T' = \kappa N$ for some κ.

Solutions

1. Page 65, Exercise 7.

(a) $\bar{\alpha} = \alpha(h)$

so $\bar{\alpha}' = h'\alpha'(h)$, by Lemma I.4.5,

so $\|\bar{\alpha}'(s)\| = |h'(s)|\,\|\alpha'(h(s))\|$ for all s.

α is a unit-speed curve, so $\|\alpha'(h(s))\| = 1$ for all s; $\bar{\alpha}$ is a unit-speed curve, so $\|\bar{\alpha}'(s)\| = 1$ for all s: thus $|h'(s)| = 1$ and so $h'(s) = \pm 1$. Since h is differentiable and has an interval as its domain, $h'(s)$ cannot be $+1$ for some values of s and -1 for other values of s. So we can write $h' = \varepsilon$, where $\varepsilon = \pm 1$ independent of s. Then $h(s) = \varepsilon s + h(0) = \pm s + s_0$ where $s_0 = h(0)$.

(b) The salient fact from part (a) is that $h'(s) = \varepsilon$.

$\bar{T}(s) = \alpha(h)'(s) = h'(s)\alpha'(h(s))$, by Lemma I.4.5,

$= \varepsilon T(h(s))$.

$\bar{T}'(s) = \varepsilon T(h)'(s) = \varepsilon h'(s)T'(h(s))$, by Exercise II.2.7,

$= \varepsilon^2 T'(h(s)) = T'(h(s))$:

thus

$$\bar{\kappa}(s)\,\bar{N}(s) = \kappa(h(s))\,N(h(s)): \qquad (*)$$

taking norms we find

$$|\bar{\kappa}(s)|\,\|\bar{N}(s)\| = |\kappa(h(s))|\,\|N(h(s))\|$$

so $|\bar{\kappa}(s)| = |\kappa(h(s))|$ since $\|\bar{N}(s)\| = \|N(h(s))\| = 1$:

$\kappa(h(s))$ and $\bar{\kappa}(s)$ are positive so we may remove the modulus signs, obtaining

$$\bar{\kappa}(s) = \kappa(h(s)).$$

Now this equation and (*) together imply $\bar{N}(s) = N(h(s))$ since $\kappa(h(s))$ is not zero.

$\bar{B}(s) = \bar{T}(s) \times \bar{N}(s) = \varepsilon T(h(s)) \times N(h(s)) = \varepsilon B(h(s))$.

$\bar{B}'(s) = \varepsilon B(h)'(s) = \varepsilon h'(s)\,B'(h(s))$, by Exercise II.2.7,

$= \varepsilon^2\,B'(h(s)) = B'(h(s)) = -\tau(h(s))\,N(h(s))$:

thus

$$-\bar{\tau}(s).\bar{N}(s) = -\tau(h(s))\, N(h(s)),$$

but

$$\bar{N}(s) = N(h(s))$$

so

$$\bar{\tau}(s) = \tau(h(s)).$$

The importance of this result is that if α and $\bar{\alpha}$ are two different unit-speed reparametrizations (in the sense of Theorem II.2.1) of some curve γ, then α and $\bar{\alpha}$ have the same Frenet apparatus at each point of the curve.

2. Page 63, Exercise 1.

$$\beta(s) = (\tfrac{4}{5}\cos s,\ 1 - \sin s,\ -\tfrac{3}{5}\cos s).$$

$$\beta'(s) = (-\tfrac{4}{5}\sin s,\ -\cos s,\ \tfrac{3}{5}\sin s)$$

$$\|\beta'(s)\| = (\tfrac{16}{25}\sin^2 s + \cos^2 s + \tfrac{9}{25}\sin^2 s)^{\frac{1}{2}} = (\cos^2 s + \sin^2 s)^{\frac{1}{2}} = 1,$$

so β is indeed a unit-speed curve and

$$T(s) = \beta'(s).$$

$$T'(s) = (-\tfrac{4}{5}\cos s,\ \sin s,\ \tfrac{3}{5}\cos s)$$

$$\kappa(s) = \|T'(s)\| = (\tfrac{16}{25}\cos^2 s + \sin^2 s + \tfrac{9}{25}\cos^2 s)^{\frac{1}{2}}$$
$$= (\cos^2 s + \sin^2 s)^{\frac{1}{2}} = 1, \text{ so}$$

$$N(s) = T'(s).$$

$$B(s) = T(s) \times N(s) = \begin{vmatrix} U_1 & U_2 & U_3 \\ -\frac{4}{5}\sin s & -\cos s & \frac{3}{5}\sin s \\ -\frac{4}{5}\cos s & \sin s & \frac{3}{5}\cos s \end{vmatrix}$$

$$= \left(-\tfrac{3}{5}(\cos^2 s + \sin^2 s),\ -\tfrac{12}{25}(\sin s\cos s - \sin s\cos s),\ -\tfrac{4}{5}(\sin^2 s + \cos^2 s)\right)$$

$$= \left(-\tfrac{3}{5}, 0, -\tfrac{4}{5}\right).$$

$B'(s) = \mathbf{0}$ so $\tau(s) = 0$.

Since $\tau = 0$ and κ is constant, Lemma II.3.6 tells us that β is part of a circle of radius $\frac{1}{\kappa} = 1$ The proof of Lemma II.3.6 showed that the centre of the circle is given by $\beta(s) + (1/\kappa(s))\, N(s)$, which in this case gives

$$(\tfrac{4}{5}\cos s,\ 1 - \sin s,\ -\tfrac{3}{5}\cos s) + (-\tfrac{4}{5}\cos s,\ \sin s,\ \tfrac{3}{5}\cos s)$$

$$= (0, 1, 0).$$

3. σ is the curve whose coordinates are the same as those of the vector field T, and so

$$\|\sigma'(s)\| = \|T'(s)\| = \kappa(s).$$

We saw in the preceding exercise that $\kappa(s) = 1$ and so $\|\sigma'(s)\| = 1$ for all s, that is, σ has unit speed.

4. Page 65, Exercise 10.

(a) If α lies on the sphere of centre c and radius r, then

$$\|\alpha(s) - c\| = r,$$

that is,

$$(\alpha(s) - c) \cdot (\alpha(s) - c) = r^2.$$

Using coordinates only and ignoring points of application (following Comment (iii)), we differentiate to obtain

$$(\alpha(s) - c) \cdot \alpha'(s) = 0,$$

that is,

$$(\alpha(s) - c) \cdot T(s) = 0.$$

By the orthonormal expansion of $\alpha(s) - c$ in terms of the basis T(s), N(s), B(s),

$$\alpha(s) - c = a(s)N(s) + b(s)B(s)$$

where

$$a(s) = (\alpha(s) - c) \cdot N(s), \; b(s) = (\alpha(s) - c) \cdot B(s).$$

Differentiating again:

$$\alpha'(s) = a'(s)N(s) + a(s)\,N'(s) + b'(s)\,B(s) + b(s)B'(s),$$

but

$$\alpha'(s) = T(s)$$

so that

$$T(s) = a'(s)N(s) + a(s)(-\kappa(s)T(s) + \tau(s)B(s)) + b'(s)B(s) - b(s)\tau(s)N(s).$$

Since T(s), N(s), B(s), form a basis, we can equate coefficients on each side: coefficients of T(s) give

$$1 = -a(s)\kappa(s)$$

and so

$$a(s) = \frac{-1}{\kappa(s)} = -\,\rho(s);$$

coefficients of N(s) give

$$0 = a'(s) - b(s)\tau(s)$$

so

$$b(s) = \frac{1}{\tau(s)}\,a'(s)$$
$$= -\,\sigma(s)\rho'(s).$$

Thus $\alpha - c = -\,\rho N - \rho'\sigma B$.

Since N and B are orthogonal,

$$\|\alpha - c\|^2 = (-\rho)^2 + (-\rho'\sigma)^2$$

so

$$r^2 = \rho^2 + (\rho'\sigma)^2.$$

(b) We put $\gamma(s) = \alpha(s) + \rho(s)N(s) + \rho'(s)\sigma(s)B(s)$ as we know by part (a) that $\gamma(s)$ is constant when α is a spherical curve. We aim to show that $\gamma(s)$ is a constant if $\rho^2 + (\rho'\sigma)^2 = r^2$. We can achieve this if we can show that $\gamma' = 0$.

Now

$$\begin{aligned}\gamma' &= \alpha' + \rho'N + \rho N' + (\rho'\sigma)'B + \rho'\sigma B' \\ &= T + \rho'N + \rho(-\kappa T + \tau B) + (\rho'\sigma)'B - \rho'\sigma\tau N \\ &= (1 - \rho\kappa)T + (\rho' - \rho'\sigma\tau)N + (\rho\tau + (\rho'\sigma)')B \\ &= (\rho\tau + (\rho'\sigma)')B \qquad \text{since } \rho\kappa = \sigma\tau = 1.\end{aligned}$$

Now we use our information that $\rho^2 + (\rho'\sigma)^2 = r^2$. Differentiation gives

$$2\rho\rho' + 2\rho'\sigma(\rho'\sigma)' = 0$$

so

$$\rho + \sigma(\rho'\sigma)' = 0, \qquad \text{because } \rho' \neq 0,$$

so

$$\rho\tau + \sigma\tau(\rho'\sigma)' = 0$$

so

$$\rho\tau + (\rho'\sigma)' = 0, \qquad \text{because } \sigma\tau = 1.$$

Thus

$$\gamma' = (\rho\tau + (\rho'\sigma)')B = 0 \qquad \text{and so } \gamma \text{ is constant.}$$

Putting $\gamma(s) = c$ gives

$$\alpha(s) - c = -\rho(s)N(s) - \rho'(s)\sigma(s)B(s)$$

and so, as before,

$$\begin{aligned}\|\alpha - c\|^2 &= (-\rho)^2 + (-\rho'\sigma)^2 \\ &= \rho^2 + (\rho'\sigma)^2 \\ &= r^2 \qquad \text{by assumption}\end{aligned}$$

so

$$\|\alpha - c\| = r,$$

that is, α lies on the sphere with centre c and radius r.

5. Page 65, Exercise 8.

(a) $N = (-y', x')$, so

$N' = (-y'', x'')$.

However, $T' = (x'', y'') = \kappa N = \kappa(-y', x')$ so

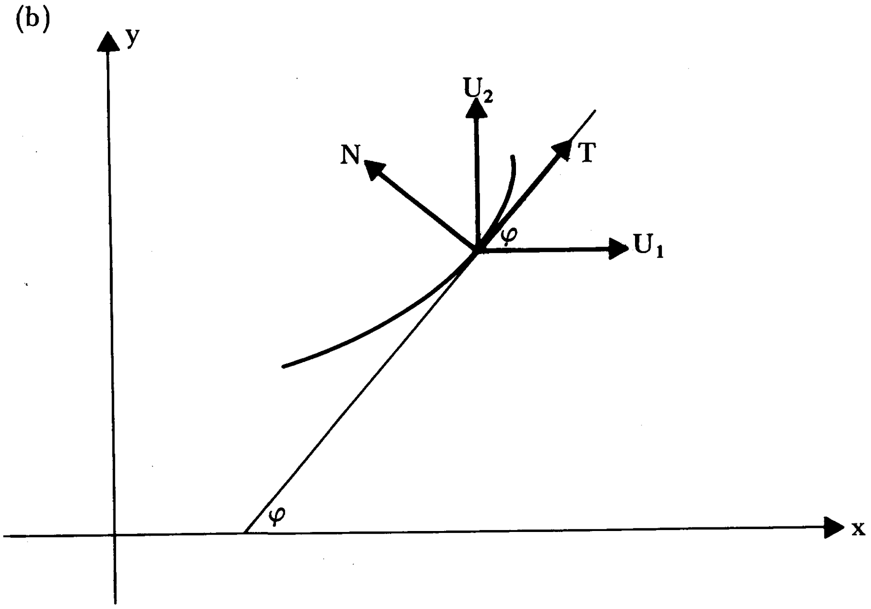
$$x'' = -\kappa y'$$

$$y'' = \kappa x'$$

and so

$$N' = (-\kappa x', -\kappa y') = -\kappa(x', y') = -\kappa T.$$

(b)



Since T has slope angle φ and unit length

$$T = \cos\varphi\, U_1 + \sin\varphi\, U_2$$

and so

$$T' = -\sin\varphi\, \varphi' U_1 + \cos\varphi\, \varphi' U_2.$$

N has slope angle $(\varphi + \frac{\pi}{2})$ and unit length, so

$$N = \cos(\varphi + \tfrac{\pi}{2}) U_1 + \sin(\varphi + \tfrac{\pi}{2}) U_2$$
$$= -\sin\varphi\, U_1 + \cos\varphi\, U_2.$$

Thus

$$T' = \varphi' N.$$

However,

$T' = \kappa N$, and so $\kappa = \varphi'$.

II.4 ARBITRARY-SPEED CURVES

Introduction

This section extends the results of Section II.3 to arbitrary speed curves. We need to use the chain rule for reparametrizations of curves (Lemma I.4.5) and its analogue for vector fields on reparametrized curves (Exercise II.2.7) to adapt the Frenet apparatus to an arbitrary speed curve. We are searching for an effective procedure for computing the Frenet apparatus of an arbitrary regular curve, and, although we succeed in this, we lose the simplicity of the earlier Frenet formulas, with the result that the theory becomes cumbersome and the computations more involved and so liable to error. Although this whole section is important and very practical, if you find yourself becoming bogged down in calculus and computation give it a rest for a while and return later: the results of this section are not needed again until Section III.5.

READ: Section II.4 (pages 66–74), omitting the last full paragraph on page 67.

Comments

(i) Page 66: the definitions The curvature, torsion etc. of α are those at the same point of the corresponding unit-speed curve $\bar{\alpha}$, that is, if p is the point $\alpha(t) = \bar{\alpha}(s(t))$ on the curve, then the curvature at p is $\kappa(t) = \bar{\kappa}(s(t))$. There is a possible difficulty here, in that if we take different unit-speed reparametrizations of α they may give a different Frenet apparatus. However, the result of Exercise II.3.7 tells us that so long as we restrict ourselves to unit-speed reparametrizations in the sense of Theorem II.2.1 the Frenet apparatus is the same. Thus "the Frenet apparatus of α is the Frenet apparatus of any unit-speed reparametrization of α" is a valid definition.

(ii) Page 73: the orthonormal expansion of **u** By Theorem II.1.5

$$\mathbf{u} = (\mathbf{u}\cdot T)T + (\mathbf{u}\cdot N)N + (\mathbf{u}\cdot B)B$$

$$= \cos\vartheta\ T + aB \text{ for some a, because } \mathbf{u}\cdot T = \cos\vartheta \text{ and } \mathbf{u}\cdot N = 0.$$

Now **u** is a unit vector, so

$$1 = \|\mathbf{u}\|^2 = \cos^2\vartheta + a^2$$

so

$$a^2 = 1 - \cos^2\vartheta = \sin^2\vartheta$$

so

$$a = \pm\sin\vartheta.$$

If $a = +\sin\vartheta$ we have the required expression; if $a = -\sin\vartheta$ then, because $\cos(-\vartheta) = \cos\vartheta$ and $\sin(-\vartheta) = -\sin\vartheta$, we can replace ϑ by $-\vartheta$ and obtain

$$\mathbf{u} = \cos(-\vartheta)T + \sin(-\vartheta)B.$$

This is possible because we are allowing ϑ to have any value. The recognition that we can just write $\sin\vartheta$ for a in such a situation should become second nature to you.

Supplementary Comments

(i) *Page 67: second paragraph* You are not seriously expected to reparametrize the curve $\alpha(t) = (t, t^2, t^3)$. If you attempted it and got stuck then you will understand O'Neill's point – there may not be explicit formulas for $\bar{\alpha}$.

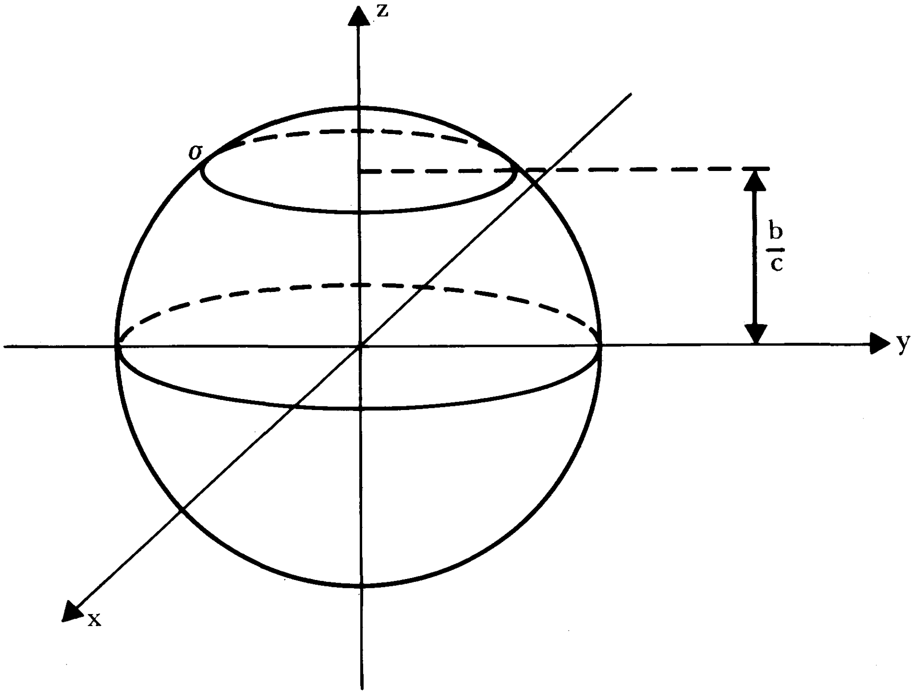
(ii) *Page 69: the formula for κ* Since $v = \|\alpha'\|$, the equation

$$\|\alpha' \times \alpha''\| = \kappa v^3$$

yields

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{v^3} = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}.$$

(iii) *Page 71: the example*



Summary

Notation

σ	Page 71, line 10

Definitions

(i)	Frenet apparatus for an arbitrary-speed curve	Page 66, first paragraph
(ii)	Spherical image σ	Page 71, line 10
(iii)	Cylindrical helix	Page 72, Definition 4.5
(iv)	Circular helix	Text: page 41, Exercise 5

Results

(i) $T' = \kappa v N$

$N' = -\kappa v T + \tau v B$

$B' = -\tau v N$ Page 67, Lemma 4.1

(ii) $\alpha' = vT$

$$\alpha'' = \frac{dv}{dt}T + \kappa v^2 N$$

Page 68, Lemma 4.2

(iii) $T = \dfrac{\alpha'}{\|\alpha'\|}$ $\qquad \kappa = \dfrac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}$

$$B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} \qquad \tau = \frac{(\alpha' \times \alpha'')\cdot\alpha'''}{\|\alpha' \times \alpha''\|^2}$$

$N = B \times T$

Page 69, Theorem 4.3

(iv) For a regular curve:

$\kappa = 0$ if and only if α is a straight line

$\tau = 0$ if and only if α is planar

κ constant >0, $\tau = 0$ if and only if α is part of a circle

κ constant >0, τ constant $\neq 0$ if and only if α is a circular helix

τ/κ constant if and only if α is a cylindrical helix.

Page 74

Techniques

(i) Finding the Frenet apparatus of an arbitrary regular curve. Page 69, Theorem 4.3

(ii) Finding the spherical image of a curve. Page 71, second paragraph

(iii) Finding the Frenet apparatus for a curve derived from some other curve in terms of the latter's Frenet apparatus: this involves repeated application of the Frenet formulas and taking higher derivatives. Page 71, last paragraph

(iv) Finding **u** and ϑ for a cylindrical helix by putting $\cot\vartheta = \tau/\kappa$ and $\mathbf{u} = \cos\vartheta\, T + \sin\vartheta\, B$. Page 73, Theorem 4.6

Exercises

Technique (i)

1. Page 74, Exercise 2. This continues the example of Exercise II.2.3, whose solution is on Text page 23.

Techniques (ii) and (iii)

2. Page 75, Exercise 11. κ_σ has been found on page 72, so all you need find is τ_σ. Make use of the working already done on pages 71–72.

Technique (iii)

3. Page 74, Exercise 8.

Technique (iv)

4. Page 75, Exercise 9(c).

Theory Exercises (omit if you are short of time)

5. Page 75, Exercise 10. This continues from Exercise II.4.8: a cylindrical helix β is a circular helix if its cross-section curve γ is part of a circle.

6. Page 75, Exercise 12(a). This continues from Exercise II.4.11.

Optional Section – Curves in the Plane

If you have time to follow this optional line of study, you should also do:

7. Let α be a plane curve, $\bar{\alpha}$ a unit-speed reparametrization of α, that is $\alpha = \bar{\alpha}(s)$ where s is an arc-length function for α. Define

   $T = \bar{T}(s)$

   $\kappa = \bar{\kappa}(s)$

   N as in Exercise 8 on page 65.

   Prove that (a) $\alpha' = vT$

   (b) $T' = v\kappa N$

   (c) $N' = -\kappa v T$.

8. Page 75, Exercise 14.

9. Page 75, Exercise 15. The central curve is defined in Exercise II.4.13. After you have sketched both curves, how do you account for the cusp (sharp point) in α^* at $\alpha^*(0)$?

Solutions

1. Page 74, Exercise 2.

   $\alpha(t) = (\cosh t, \sinh t, t)$.

   We already know that:

   $\alpha'(t) = (\sinh t, \cosh t, 1)$

   $v(t) = \sqrt{2} \cosh t$

   $s(t) = \sqrt{2} \sinh t$.

   Now

   $\alpha''(t) = (\cosh t, \sinh t, 0)$

   $\alpha'''(t) = (\sinh t, \cosh t, 0)$

   so $\alpha'(t) \times \alpha''(t) = (-\sinh t, \cosh t, \sinh^2 t - \cosh^2 t) = (-\sinh t, \cosh t, -1)$,

   $\|\alpha'(t) \times \alpha''(t)\| = (\sinh^2 t + \cosh^2 t + 1)^{\frac{1}{2}} = \sqrt{2} \cosh t$,

   $\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t) = -\sinh^2 t + \cosh^2 t = 1$.

Now we simply apply Theorem II.4.3:

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|} = \frac{1}{\sqrt{2}}(\tanh t, 1, \operatorname{sech} t) \text{ (writing tanh for } \frac{\sinh}{\cosh} \text{ and sech for } \frac{1}{\cosh})$$

$$B(t) = \frac{\alpha'(t) \times \alpha''(t)}{\|\alpha'(t) \times \alpha''(t)\|} = \frac{1}{\sqrt{2}}(-\tanh t, 1, -\operatorname{sech} t)$$

$$N(t) = B(t) \times T(t) = \frac{1}{2}(2\operatorname{sech} t, 0, -2\tanh t) = (\operatorname{sech} t, 0, -\tanh t)$$

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3} = \frac{\sqrt{2}\cosh t}{(\sqrt{2}\cosh t)^3} = \frac{1}{2\cosh^2 t}$$

$$\tau(t) = \frac{\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2} = \frac{1}{(\sqrt{2}\cosh t)^2} = \frac{1}{2\cosh^2 t}$$

Since we know $s(t) = \sqrt{2}\sinh t$, $2\cosh^2 t = 2 + 2\sinh^2 t = 2 + s^2$,

so $\kappa(s) = \tau(s) = \dfrac{1}{2 + s^2}$.

2. Page 75, Exercise 11.

Let β have Frenet apparatus T, N, B, κ, τ.

Making the usual identification of coordinates and ignoring both the distinction between points and tangent vectors and the points of application of tangent vectors, we can write, as on pages 71–72:

$$\sigma = T$$

$$\sigma' = \kappa N$$

$$\sigma'' = -\kappa^2 T + \kappa' N + \kappa\tau B$$

$$\sigma' \times \sigma'' = \kappa^2(\kappa B + \tau T)$$

$$\|\sigma' \times \sigma''\| = \kappa^2(\kappa^2 + \tau^2)^{\frac{1}{2}}.$$

Now

$$\begin{aligned}\sigma''' &= -2\kappa\kappa' T - \kappa^2 T' + \kappa'' N + \kappa' N' + \kappa'\tau B + \kappa\tau' B + \kappa\tau B' \\ &= -2\kappa\kappa' T - \kappa^3 N + \kappa'' N - \kappa'\kappa T + \kappa'\tau B + \kappa'\tau B + \kappa\tau' B - \kappa\tau^2 N \\ &= -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (\kappa\tau' + 2\kappa'\tau)B\end{aligned}$$

so

$$\begin{aligned}\sigma' \times \sigma'' \cdot \sigma''' &= \kappa^2\tau(-3\kappa\kappa') + \kappa^3(\kappa\tau' + 2\kappa'\tau) \\ &= \kappa^3(\kappa\tau' - \tau\kappa') \\ &= \kappa^5(\tau/\kappa)'.\end{aligned}$$

$$\tau_\sigma = \frac{\sigma' \times \sigma'' \cdot \sigma'''}{\|\sigma' \times \sigma''\|^2} = \frac{\kappa^5(\tau/\kappa)'}{\kappa^4(\kappa^2 + \tau^2)} = \frac{(\tau/\kappa)'}{\kappa(1 + (\tau/\kappa)^2)}$$

3. Page 74, Exercise 8.

(a) We have to show that $f(t) = (\gamma(t) - \alpha(0))\cdot\mathbf{u} = 0$.

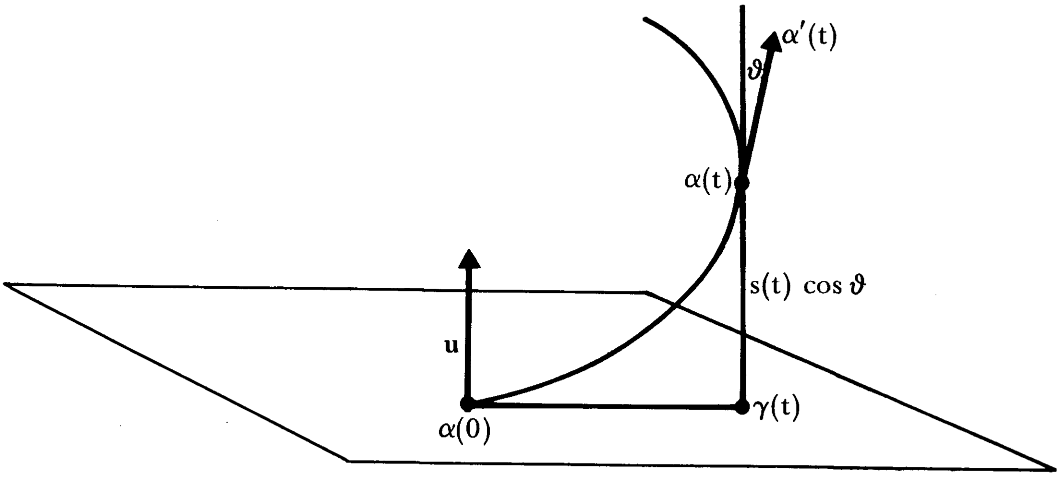
We know that

$$f(0) = (\gamma(0) - \alpha(0))\cdot\mathbf{u} = s(0)\cos\vartheta\ \mathbf{u}\cdot\mathbf{u} = 0$$

because $s(0) = 0$, and so it is sufficient to show that f is a constant function, i.e. that $f' = 0$.

$$\begin{aligned} f(t) &= (\gamma(t) - \alpha(0))\cdot\mathbf{u} \\ &= (\alpha(t) - s(t)\cos\vartheta\,\mathbf{u} - \alpha(0))\cdot\mathbf{u} \\ &= (\alpha(t) - \alpha(0))\cdot\mathbf{u} - s(t)\cos\vartheta \qquad \text{because } \mathbf{u}\cdot\mathbf{u} = 1; \\ f'(t) &= \alpha'(t)\cdot\mathbf{u} - s'(t)\cos\vartheta \\ &= v(t)\,T(t)\cdot\mathbf{u} - v(t)\cos\vartheta \\ &= v(t)\cos\vartheta - v(t)\cos\vartheta \\ &= 0. \end{aligned}$$

Hence $\gamma(t) - \alpha(0)$ is always perpendicular to $\mathbf{u}$, that is, γ lies in the plane through $\alpha(0)$ orthogonal to $\mathbf{u}$.



(b) Let us suppose that α has unit-speed, so that $s(t) = t$ and $v(t) = 1$. Using bars to denote the Frenet apparatus of γ:

$$\gamma(t) = \alpha(t) - t\cos\vartheta\,\mathbf{u}$$

$$\gamma'(t) = \alpha'(t) - \cos\vartheta\,\mathbf{u} = T(t) - \cos\vartheta\,\mathbf{u}. \qquad (*)$$

$$\begin{aligned} \bar{v} &= \|\gamma'\| \\ &= ((T - \cos\vartheta\,\mathbf{u})\cdot(T - \cos\vartheta\,\mathbf{u}))^{\frac{1}{2}} \\ &= (T\cdot T - 2\cos\vartheta\, T\cdot\mathbf{u} + \cos^2\vartheta\,\mathbf{u}\cdot\mathbf{u})^{\frac{1}{2}} \\ &= (1 - 2\cos^2\vartheta + \cos^2\vartheta)^{\frac{1}{2}} \\ &= (1 - \cos^2\vartheta)^{\frac{1}{2}} \\ &= (\sin^2\vartheta)^{\frac{1}{2}} = |\sin\vartheta|. \end{aligned}$$

Since $\bar{v}$ is constant, Lemma II.4.2 gives:

$$\begin{aligned} \gamma''(t) &= \bar{\kappa}(t)\,(\bar{v}(t))^2\,\bar{N}(t) \\ &= \bar{\kappa}(t)\sin^2\vartheta\,\bar{N}(t), \end{aligned}$$

but our first expression, (*), for $\gamma'(t)$ gives:

$$\gamma''(t) = T'(t) = \kappa(t)N(t),$$

so $\bar{\kappa}(t)\sin^2\vartheta\,\bar{N}(t) = \kappa(t)N(t)$.

Taking norms gives $\bar{\kappa}(t) = (\kappa(t)/\sin^2\vartheta)$ since $\kappa(t), \bar{\kappa}(t)$, and $\sin^2\vartheta$ are all positive.

4. Page 75, Exercise 9(c).

We use the technique used in the proof of Theorem II.4.6. We already have all the necessary information about α (see solution to Exercise II.4.2).

$\tau(t) = \kappa(t) = \dfrac{1}{2\cosh^2 t}$, so $\tau(t)/\kappa(t) = 1$: we choose

$$\vartheta = \frac{\pi}{4} \text{ so that } \cot\vartheta = 1 = \frac{\tau(t)}{\kappa(t)}.$$

Now we put $U(t) = \cos\frac{\pi}{4}\,T(t) + \sin\frac{\pi}{4}\,B(t)$

$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\,(\tanh t,\ 1,\ \operatorname{sech} t) + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\,(-\tanh t,\ 1,\ -\operatorname{sech} t)$$

$$= (0, 1, 0).$$

This vector field U is parallel, as expected, so we put $\mathbf{u} = (0, 1, 0)$, and $\mathbf{u}$ is a fixed unit vector such that $T(t)\cdot\mathbf{u} = \cos(\pi/4)$.

$$\begin{aligned}\gamma(t) &= \alpha(t) - s(t)\cos\frac{\pi}{4}\,\mathbf{u} \\ &= (\cosh t,\ \sinh t,\ t) - \sqrt{2}\sinh t.\ \frac{1}{\sqrt{2}}\,(0, 1, 0) \\ &= (\cosh t,\ 0,\ t).\end{aligned}$$

$$\gamma(t) - \alpha(0) = (\cosh t,\ 0,\ t) - (1, 0, 0) = (\cosh t - 1, 0, t)$$

so $(\gamma(t) - \alpha(0))\cdot\mathbf{u} = (\cosh t - 1, 0, t)\cdot(0, 1, 0) = 0$, as required.

5. Page 75, Exercise 10.

Since $\kappa > 0$ and $\tau \neq 0$ are constants, we can apply Theorem II.4.6: thus β is certainly a cylindrical helix with

$$\vartheta = \cot^{-1}(\tau/\kappa), \qquad \mathbf{u} = \cos\vartheta\,T(t) + \sin\vartheta\,B(t).$$

All that remains to show is that the cross-section curve γ of β is part of a circle. By Exercise II.4.8(a), γ is planar; by Exercise II.4.8(b) the curvature of γ is $(\kappa/\sin^2\vartheta)$, which is a constant: hence, by Lemma II.3.6 (which is unaltered if the unit-speed restriction is lifted), γ is part of a circle.

6. Page 75, Exercise 12(a).

We may suppose the curve β has unit-speed. Let σ be its spherical image.

β is a cylindrical helix $\Longleftrightarrow$ τ/κ is constant (by Theorem II.4.6)

$\Longleftrightarrow$ κ_σ is constant and τ_σ is zero (by Exercise II.4.11)

$\Longleftrightarrow$ σ is part of a circle (by Lemma II.3.6 and its converse).

7. (a) $\alpha = \bar{\alpha}(s)$

so

$$\begin{aligned} \alpha' &= s'\bar{\alpha}'(s), && \text{by Lemma I.4.5,} \\ &= v\bar{\alpha}'(s), && \text{because } s' = v, \\ &= v\bar{T}(s) \\ &= vT. \end{aligned}$$

(b) $T = \bar{T}(s)$

so

$$\begin{aligned} T' &= s'\bar{T}'(s), && \text{by Exercise II.2.7,} \\ &= v\bar{T}'(s) \\ &= v\bar{\kappa}(s)\bar{N}(s), && \text{because } \bar{T}' = \bar{\kappa}\bar{N}, \\ &= v\kappa N. \end{aligned}$$

(c) $N = \bar{N}(s)$

so

$$\begin{aligned} N' &= s'\bar{N}'(s), && \text{by Exercise II.2.7,} \\ &= v\bar{N}'(s) \\ &= v(-\bar{\kappa}(s)\bar{T}(s)), && \text{by Exercise II.3.8,} \\ &= -v\kappa T. \end{aligned}$$

8. Page 75, Exercise 14.

$$\begin{aligned} \alpha' &= vT \\ \alpha'' &= v'T + vT' \\ &= v'T + v^2\kappa N \end{aligned}$$

Thus $\alpha''\cdot N = v^2\kappa$.

However, in the plane $N = J(T) = \dfrac{J(\alpha')}{v}$, so

$$\alpha''\cdot\frac{J(\alpha')}{v} = v^2\kappa$$

i.e.

$$\kappa = \frac{\alpha''\cdot J(\alpha')}{v^3}.$$

In terms of the coordinate functions,

$$\begin{aligned} \alpha' &= (x', y')_\alpha \\ J(\alpha') &= (-y', x')_\alpha \\ v &= (x'^2 + y'^2)^{\frac{1}{2}} \\ \alpha'' &= (x'', y'')_\alpha \end{aligned}$$

so $$\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}}.$$

9. Page 75, Exercise 15.

(a) $\alpha^* = \alpha + \frac{1}{\kappa} N$

$$\alpha^{*\prime} = \alpha' + \frac{1}{\kappa} N' - \frac{\kappa'}{\kappa^2} N$$

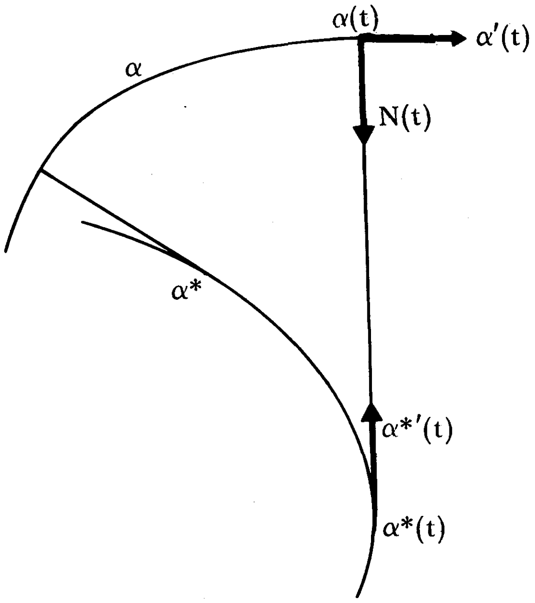
$$= vT - \frac{v\kappa}{\kappa} T - \frac{\kappa'}{\kappa^2} N$$

$$= -\frac{\kappa'}{\kappa^2} N.$$

The tangent line to α^* at $\alpha^*(t)$ is the line through $\alpha^*(t)$ in the direction $\alpha^{*\prime}(t)$; the normal line to α at $\alpha(t)$ is the line through $\alpha(t)$ in the direction $N(t)$, that is in the direction orthogonal to $\alpha'(t)$. Consequently these two lines are the same if and only if

(i) $\alpha^{*\prime}(t)$ and $\alpha'(t)$ are orthongonal: $\alpha^{*\prime}(t)\cdot\alpha'(t) = 0$

(ii) there is a number $u(t)$ such that $\alpha^*(t) = \alpha(t) + u(t)N(t)$.



Firstly, let us check that α^* does satisfy these conditions.

$$\alpha^{*\prime}(t)\cdot\alpha'(t) = \frac{-\kappa'(t)}{(\kappa(t))^2} N(t)\cdot\alpha'(t) = 0$$

so (i) is satisfied. Moreover

$$\alpha^*(t) = \alpha(t) + \frac{1}{\kappa(t)} N(t)$$

so condition (ii) is satisfied with $u(t) = \frac{1}{\kappa(t)}$.

Conversely, suppose β is a curve satisfying (i) and (ii), that is

$$\beta'\cdot\alpha' = 0$$

and

$$\beta = \alpha + uN.$$

Then

$$\begin{aligned}\beta' &= \alpha' + u'N + uN'\\ &= vT + u'N - uv\kappa T\\ &= v(1 - u\kappa)T + u'N\end{aligned}$$

so

$$\begin{aligned}\beta'\cdot\alpha' &= (v(1 - u\kappa)T + u'N)\cdot(vT)\\ &= v^2(1 - u\kappa).\end{aligned}$$

$\beta'\cdot\alpha' = 0$ and v is non-zero, so this implies $\kappa = \frac{1}{u}$, that is

$$\beta = \alpha + \frac{1}{\kappa}N = \alpha^*.$$

(b) We have already shown that $N = \frac{J(\alpha')}{v}$, $\kappa = \frac{\alpha''\cdot J(\alpha')}{v^3}$,

$$\begin{aligned}\text{so } \alpha^* &= \alpha + \frac{1}{\kappa}N\\ &= \alpha + \frac{v^3}{\alpha''\cdot J(\alpha')}\frac{J(\alpha')}{v}\\ &= \alpha + \frac{v^2}{\alpha''\cdot J(\alpha')}J(\alpha')\\ &= \alpha + \frac{\alpha'\cdot\alpha'}{\alpha''\cdot J(\alpha')}J(\alpha').\end{aligned}$$

(c) $\alpha(t) = (t + \sin t, 1 + \cos t) \qquad -\pi < t < \pi$

$\alpha'(t) = (1 + \cos t, -\sin t)$

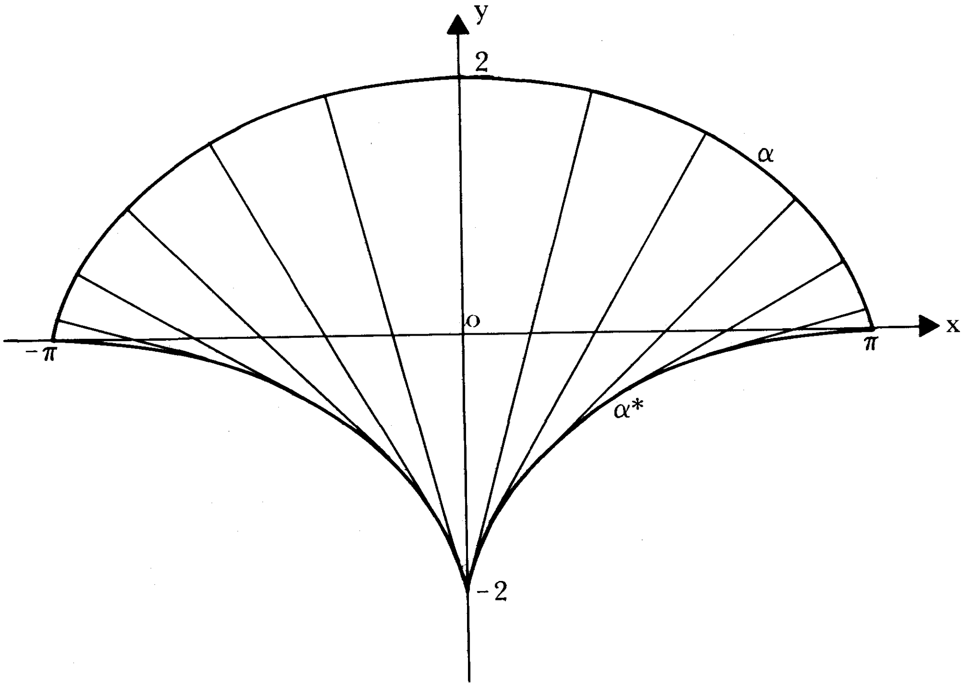
$\alpha''(t) = (-\sin t, -\cos t)$

$\alpha'(t)\cdot\alpha'(t) = 1 + 2\cos t + \cos^2 t + \sin^2 t = 2 + 2\cos t$

$J(\alpha'(t)) = (\sin t, 1 + \cos t)$

$\alpha''(t)\cdot J(\alpha'(t)) = -\sin^2 t - \cos t - \cos^2 t = -1 - \cos t.$

$$\begin{aligned}\text{Now } \alpha^*(t) &= \alpha(t) + \frac{\alpha'(t)\cdot\alpha'(t)}{\alpha''(t)\cdot J(\alpha'(t))}J(\alpha'(t))\\ &= \alpha(t) - 2J(\alpha'(t))\\ &= (t + \sin t, 1 + \cos t) - 2(\sin t, 1 + \cos t)\\ &= (t - \sin t, -1 - \cos t).\end{aligned}$$



The cusp at $\alpha^*(0)$ suggests that α^* is not regular at that point. In fact that is the case, for

$$\alpha^{*\prime}(t) = (1 - \cos t,\ \sin t)$$

so

$$\alpha^{*\prime}(0) = (0, 0).$$

II.5 COVARIANT DERIVATIVES

Introduction

This section depends on Sections I.3 and II.2; you also need to be familiar with the differential of a function from E^3 to R (Section I.5). We introduce a new derivative, the *covariant derivative*. This is a very important idea in differential geometry, and serves as the starting point for a lot of more advanced work. The definition is similar to the other definitions of derivatives in the book, in that it is given in terms of a derivative we already know along the particular curve $t \longmapsto p + tv$. It generalizes the derivative of a vector field on a curve.

READ: Section II. 5 (pages 77-80).

Comment

(i) *Page 78: the symbol* ∇ This is usually pronounced "del" or "nabla".

Additional Text

(i) *Covariant Differential* If $W = \Sigma w_i U_i$ is a vector field on E^3, the *covariant differential*, ∇W, of W is the function on tangent vectors defined by

$$\nabla W(v) = \nabla_v W.$$

By Lemma II.5.2 $\nabla_v W = \Sigma v[w_i]\, U_i(p)$ and so

$$\begin{aligned}\nabla W(v) &= \Sigma v[w_i]\, U_i(p)\\ &= \Sigma dw_i(v) U_i(p), \qquad \text{by definition of } dw_i,\\ &= (\Sigma dw_i U_i)(v).\end{aligned}$$

Thus we may write

$$\nabla W = \Sigma dw_i U_i.$$

(ii) *A Useful Result* The result of the following exercise will be used often later in the course. Attempt the exercise now.

1. Page 81, Exercise 6(a). (HINT: Use Lemma I.4.6.)

Summary

Notation

$\nabla_v W$	Page 78, line 2
$\nabla_V W$	Page 79, line 15
∇W	Text, page 49

Definitions

(i)	Covariant derivative with respect to a tangent vector $\nabla_v W$	Page 77, Definition 5.1
(ii)	Covariant derivative with respect to a vector field $\nabla_V W$	Page 79, line 12
(iii)	Covariant differential ∇W	Text, page 49

Results

(i)	If $W = \Sigma w_i U_i$, $\nabla_v W = \Sigma v[w_i]\, U_i(p)$	Page 78, Lemma 5.2
	$\nabla_V W = \Sigma V[w_i]\, U_i$.	Page 79, line 18
(ii)	Linearity and Leibnizian properties of the covariant derivative.	
	With respect to a tangent vector v:	
	$\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$	
	$\nabla_v(aY + bZ) = a\nabla_v Y + b\nabla_v Z$	
	$\nabla_v(fY) = v[f]\, Y(p) + f(p)\nabla_v Y$	
	$v[Y\cdot Z] = \nabla_v Y\cdot Z(p) + Y(p)\cdot\nabla_v Z$	Page 78, Theorem 5.3
	With respect to a vector field V:	
	$\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_W Y$	
	$\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$	
	$\nabla_V(fY) = V[f]\, Y + f\nabla_V Y$	
	$V[Y\cdot Z] = \nabla_V Y\cdot Z + Y\cdot\nabla_V Z$	Page 80, Corollary 5.4
(iii)	If $W = \Sigma w_i U_i$, $\nabla W = \Sigma\, dw_i U_i$.	Text, page 49
(iv)	$\nabla_{\alpha'(t)} W = (W(\alpha))'(t)$.	Page 81, Exercise 6(a)

Techniques

(i)	Finding the covariant derivative with respect to a tangent vector or vector field,		
	(a)	from first principles;	Page 77, Definition 5.1
	(b)	using result (i) and calculating the directional derivatives from first principles;	Page 78, Lemma 5.2
	(c)	using result (i), the linearity and Leibnizian properties of both the directional and covariant derivative, and the results $U_i[f] = (\partial f/\partial x_i)$, $v[f] = \Sigma v_i(\partial f/\partial x_i)$;	Page 78, Lemma 5.2, Page 78, Theorem 5.3 and Page 80, Corollary 5.4
	(d)	by means of the covariant differential.	Text, page 49

Exercises

Technique (i)(a)

2. Page 80, Exercise 1(a). Work directly from Definition II.5.1.

Technique (i)(b)

3. Page 80, Exercise 1(a). Use the result of Lemma II.5.2 but calculate the directional derivatives from first principles.

Technique (i)(c)

4. Page 80, Exercise 1(a). Now use all the results available to simplify the work!

5. Page 80, Exercise 2(a).

6. Page 80, Exercise 2(c).

Technique (i)(d)

7. Page 80, Exercise 5.

Solutions

1. Page 81, Exercise 6(a).

Let $W = \Sigma w_i U_i$.

$$\begin{aligned}
\text{Then } \nabla_{\alpha'(t)} W &= \sum_i \alpha'(t)[w_i]\, U_i(\alpha(t)), && \text{by Lemma II.5.2,}\\
&= \sum_i \frac{d(w_i(\alpha))}{dt}(t) U_i(\alpha(t)), && \text{by Lemma I.4.6,}\\
&= \Big(\sum_i w_i(\alpha(t)) U_i(\alpha(t))\Big)', && \text{by the definition of differentiation of a vector field on a curve,}\\
&= (W_\alpha)'(t).
\end{aligned}$$

2. Page 80, Exercise 1(a).

$p = (1, 3, -1)$, $v = (1, -1, 2)$, so

$p + tv = (1 + t, 3 - t, -1 + 2t)$.

$W = x^2 U_1 + y U_2$

so $W(p + tv) = (1 + t)^2 U_1(p + tv) + (3 - t) U_2(p + tv)$.

$$\begin{aligned}
\nabla_v W &= W(p + tv)'(0)\\
&= [2(1 + t) U_1(p + tv) - U_2(p + tv)]\big|_{t=0}\\
&= 2U_1(p) - U_2(p)\\
&= (2, -1, 0)_p.
\end{aligned}$$

3. Page 80, Exercise 1(a).

$$W = x^2U_1 + yU_2$$

so

$$\nabla_v W = v[x^2]\,U_1(p) + v[y]\,U_2(p) \qquad \text{by Lemma II.5.2.}$$

$$\begin{aligned} v[x^2] &= \frac{d}{dt}x^2(p + tv)\Big|_{t=0} \\ &= \frac{d}{dt}(1 + t)^2\Big|_{t=0} \\ &= 2(1 + t)\Big|_{t=0} \\ &= 2; \\ v[y] &= \frac{d}{dt}y(p + tv)\Big|_{t=0} \\ &= \frac{d}{dt}(3 - t)\Big|_{t=0} \\ &= -1\Big|_{t=0} \\ &= -1. \end{aligned}$$

Thus

$$\begin{aligned} \nabla_v W &= 2U_1(p) - U_2(p) \\ &= (2, -1, 0)_p. \end{aligned}$$

4. Page 80, Exercise 1(a).

$$\nabla_v W = v[x^2]\,U_1(p) + v[y]\,U_2(p).$$

$$\begin{aligned} v[x^2] &= v_1\frac{\partial(x^2)}{\partial x}(p) + v_2\frac{\partial(x^2)}{\partial y}(p) + v_3\frac{\partial(x^2)}{\partial z}(p) \\ &= v_1\ 2x(p) \\ &= 1.2.1 = 2; \\ v[y] &= v_1\frac{\partial y}{\partial x}(p) + v_2\frac{\partial y}{\partial y}(p) + v_3\frac{\partial y}{\partial z}(p) \\ &= v_2 \\ &= -1, \end{aligned}$$

so

$$\begin{aligned} \nabla_v W &= 2U_1(p) - U_2(p) \\ &= (2, -1, 0)_p. \end{aligned}$$

5. Page 80, Exercise 2(a).

$$W = \cos x\ U_1 + \sin x\ U_2$$

so

$$\begin{aligned}
\nabla_V W &= V[\cos x]\,U_1 + V[\sin x]\,U_2 && \text{by Lemma II.5.2.}\\
V[\cos x] &= (-yU_1 + xU_3)[\cos x] \\
&= -y\ U_1[\cos x] + xU_3[\cos x], && \text{by linearity,}\\
&= -y\,\frac{\partial(\cos x)}{\partial x} + x\,\frac{\partial(\cos x)}{\partial z} \\
&= y \sin x + 0; \\
V[\sin x] &= (-yU_1 + xU_3)[\sin x] \\
&= -yU_1[\sin x] + xU_3[\sin x], && \text{by linearity,}\\
&= -y\,\frac{\partial(\sin x)}{\partial x} + x\,\frac{\partial(\sin x)}{\partial z} \\
&= -y \cos x + 0.
\end{aligned}$$

Thus

$$\nabla_V W = y \sin x\ U_1 - y \cos x\ U_2.$$

6. Page 80, Exercise 2(c).

$$\begin{aligned}
\nabla_V(z^2W) &= V[z^2]W + z^2\nabla_V W && \text{by Corollary II.5.4.}\\
V[z^2] &= 2zV[z], && \text{by Corollary I.3.4,}\\
&= 2z(-yU_1 + xU_3)[z] \\
&= -2yz\ U_1[z] + 2xz\ U_3[z], && \text{by linearity,}\\
&= -2yz.0 + 2xz.1 \\
&= 2xz,
\end{aligned}$$

so

$$\begin{aligned}
\nabla_V(z^2W) &= 2xz\ W + z^2\nabla_V W \\
&= 2xz\ (\cos x\ U_1 + \sin x\ U_2) + z^2(y \sin x\ U_1 - y \cos x\ U_2) \\
&= (2xz \cos x + yz^2 \sin x)U_1 + (2xz \sin x - yz^2 \cos x)U_2.
\end{aligned}$$

7. Page 80, Exercise 5.

$$W = xy^3U_1 - x^2z^2U_3$$

so

$$\begin{aligned}
\nabla W &= d(xy^3)U_1 - d(x^2z^2)U_3 \\
&= (y^3dx + 3xy^2dy)U_1 - (2xz^2dx + 2x^2z\ dz)U_3.
\end{aligned}$$

(a) $(y^3dx + 3xy^2dy)\ ((1, 0, -3)_{(-1,\ 2,\ -1)}) = 2^3.1 + 3.(-1).2^2.0 = 8$

$(2xz^2\,dx + 2x^2z\,dz)\ ((1, 0, -3)_{(-1,\ 2,\ -1)}) = 2.(-1).(-1)^2.1 + 2.(-1)^2.(-1).(-3) = 4$

and so

$$\begin{aligned}
\nabla W\,((1, 0, -3)_{(-1,\ 2,\ -1)}) &= 8U_1(p) - 4U_3(p) \\
&= (8, 0, -4)_p.
\end{aligned}$$

(b) $(y^3dx + 3xy^2dy)\,((-1, 2, -1)_{(1,\,3,\,2)}) = 3^3.(-1) + 3.1.3^2.2 = 27$

$(2xz^2dx + 2x^2z\,dz)\,((-1, 2, -1)_{(1,\,3,\,2)}) = 2.1.2^2(-1) + 2.1^2.2.(-1) = -12$

and so

$$\begin{aligned}\nabla W((-1, 2, -1)_{(1,\,3,\,2)}) &= 27U_1(p) + 12U_3(p)\\ &= (27, 0, 12)_p.\end{aligned}$$

II.6 FRAME FIELDS

Introduction

This section does not depend on the previous two, but on Sections II.1 and II.3. The idea of a general frame field (as opposed to the Frenet frame field of Section II.3) is formulated.

READ: Section II.6 (pages 81–84).

Additional Text

(i) *The Pointwise Principle* In this section O'Neill has used the pointwise principle to extend some familiar definitions and theorems from the set of all tangent vectors to E^3 to the set S of all vector fields V in E^3. We recall that S is only a three-dimensional *module* (see Comment (ii) to Section I.2 (Text, page 10)) and so S is not isomorphic to E^3. However, because each element of S can be expressed as $\Sigma f_i U_i$, many of the ideas do carry over to S via the pointwise principle with no trouble. Some examples are given below.

(a) The vector fields E_1, E_2, E_3 form a basis for S if the tangent vectors $E_1(p)$, $E_2(p)$, $E_3(p)$ form a basis for $T_p(E^3)$ for each point p of E^3.

(b) The dot and cross products and norm are defined pointwise. That is, if V and W are vector fields then V·W is the function from E^3 to R defined by

$$(V \cdot W)(p) = V(p) \cdot W(p),$$

V × W is the vector field defined by

$$(V \times W)(p) = V(p) \times W(p),$$

and $\|V\|$ is the function from E^3 to R defined by

$$\|V\|(p) = \|V(p)\|.$$

(c) Vector fields V and W are said to be orthogonal if $V(p) \cdot W(p) = 0$ for all p. For example, V × W is orthogonal to V and W.

(d) Similarly the vector fields E_1, E_2, E_3 are orthonormal if

$$E_i \cdot E_j = \delta_{ij},$$

that is $(E_i \cdot E_i)(p) = 1$ for all p and for i = 1, 2, 3 and E_i, E_j are orthogonal for $i \neq j$.

(e) (Lemma 6.3). If E_1, E_2, E_3 are orthonormal vector fields, any vector field V has an orthonormal expansion of the form

$$\begin{aligned} V &= (V \cdot E_1)E_1 + (V \cdot E_2)E_2 + (V \cdot E_3)E_3 \\ &= f_1E_1 + f_2E_2 + f_3E_3 \end{aligned}$$

where $f_i = V \cdot E_i$.

Moreover, if $W = g_1E_1 + g_2E_2 + g_3E_3$, we find, as one would expect, that

$$V \cdot W = f_1g_1 + f_2g_2 + f_3g_3.$$

In future we shall not always trouble to point out when the pointwise principle is used to make definitions or deduce results about vector fields. So long as you are sufficiently familiar with the linear algebra of E^3 you should have no trouble with adapting this to vector fields.

Summary

Notation

$V \cdot W$	Page 82, line 1
$V \times W$	Text, page 55
$\|V\|$	Page 82, line 2
E_1, E_2, E_3	Page 82, Definition 6.1
r, ϑ, z	Page 82, Example 6.2.(1)
ρ, ϑ, φ	Page 83, Example 6.2.(2)

Definitions

(i)	Orthogonal vector fields	Text, page 55
(ii)	Orthonormal vector fields	Text, page 55
(iii)	Frame field E_1, E_2, E_3	Page 82, Definition 6.1
(iv)	Coordinate functions with respect to a frame field	Page 83, Lemma 6.3

Examples

(i)	The natural frame field U_1, U_2, U_3	Page 82, second paragraph
(ii)	The cylindrical frame field E_1, E_2, E_3	Page 82, Example 6.2.(1)
(iii)	The spherical frame field F_1, F_2, F_3	Page 83, Example 6.2.(2)

Results

(i) For the cylindrical frame field

$$E_1 = \cos\vartheta \; U_1 + \sin\vartheta \; U_2$$

$$E_2 = - \sin\vartheta \; U_1 + \cos\vartheta \; U_2$$

$$E_3 = U_3 .$$

Page 82, Example 6.2.(1)

(ii) For the spherical frame field

$$F_1 = \cos\varphi \; \cos\vartheta \; U_1 + \cos\varphi \; \sin\vartheta \; U_2 + \sin\varphi \; U_3$$

$$F_2 = -\sin\vartheta \, U_1 + \cos\vartheta \, U_2$$

$$F_3 = -\sin\varphi \cos\vartheta \, U_1 - \sin\varphi \sin\vartheta \, U_2 + \cos\varphi \, U_3 .$$

Page 83, Example 6.2.(2)

(iii) If E_1, E_2, E_3 is a frame field and V is a vector field, then

$$V = (V \cdot E_1)E_1 + (V \cdot E_2)E_2 + (V \cdot E_3)E_3 .$$

Page 83, Lemma 6.3.(1)

(iv) If E_1, E_2, E_3 is a frame field, $V = \Sigma f_i E_i$ and $W = \Sigma g_i E_i$, then

$$V \cdot W = \Sigma f_i g_i .$$

Page 83, Lemma 6.3.(2)

Techniques

(i)	Recognition of frame fields.	Page 82, Definition 6.1
(ii)	Production of frame fields using Gram-Schmidt and cross product (see Lemma II.1.A).	
(iii)	Expression of a vector field in terms of a frame field, and applications thereof.	Page 82, Example 6.2

Exercises

Techniques (i) and (ii)

1. Page 84, Exercise 1.

Technique (iii)

2. Page 84, Exercise 2.

Solutions

1. Page 84, Exercise 1.

First we notice that $\|V\|$ and $\|\widetilde{W}\|$ are never zero and so E_1 and E_2 are well-defined. Since V and W are linearly independent at each p, we know that

$$V(p) \neq 0$$
$$W(p) \neq 0$$
$$W(p) \neq \frac{(W\cdot V)(p)}{\|V(p)\|^2} V(p).$$

$V(p) \neq 0$ and so $\|V\|$ is never zero.

$$\widetilde{W} = W - (W\cdot E_1)E_1$$
$$= W - \frac{(W\cdot V)V}{\|V\|^2},$$

which is also never zero.

We must show that $E_i\cdot E_j = \delta_{ij}$.

$$E_1\cdot E_1 = \frac{V}{\|V\|}\cdot\frac{V}{\|V\|} = \frac{V\cdot V}{\|V\|^2} = \frac{\|V\|^2}{\|V\|^2} = 1;$$

$$E_2\cdot E_2 = \frac{\widetilde{W}}{\|\widetilde{W}\|}\cdot\frac{\widetilde{W}}{\|\widetilde{W}\|} = 1 \quad \text{similarly.}$$

$$E_1\cdot E_2 = \frac{V}{\|V\|}\cdot\frac{\widetilde{W}}{\|\widetilde{W}\|} = \frac{V\cdot\widetilde{W}}{\|V\|\,\|\widetilde{W}\|} = \frac{V\cdot(W - W\cdot E_1\, E_1)}{\|V\|\,\|\widetilde{W}\|}$$
$$= \frac{V\cdot W - (W\cdot E_1)(V\cdot E_1)}{\|V\|\,\|\widetilde{W}\|} = \frac{\|V\|(E_1\cdot W) - (W\cdot E_1)(\|V\|E_1\cdot E_1)}{\|V\|\,\|\widetilde{W}\|}$$
$$= \frac{\|V\|\,(W\cdot E_1) - \|V\|(W\cdot E_1)}{\|V\|\,\|\widetilde{W}\|} = 0.$$

Since $E_3 = E_1 \times E_2$, $E_1\cdot E_3 = E_2\cdot E_3 = 0$ and

$$E_3\cdot E_3 = ((E_1\cdot E_1)(E_2\cdot E_2) - (E_1\cdot E_2)^2)^{\frac{1}{2}}, \quad \text{by Lemma II.1.8,}$$
$$= 1.$$

2. Page 84, Exercise 2.

(a) $U_1 = \cos\vartheta\ E_1 - \sin\vartheta\ E_2$

$$= \cos\varphi\cos\vartheta\ F_1 - \sin\vartheta\ F_2 - \sin\varphi\cos\vartheta\ F_3.$$

(b) $\cos\vartheta\ U_1 + \sin\vartheta\ U_2 + U_3 = E_1 + E_3$

$$= (\cos\varphi + \sin\varphi)F_1 + (\cos\varphi - \sin\varphi)F_3.$$

(c) (i) $$\begin{aligned} xU_1 + yU_2 + zU_3 &= r\cos\vartheta\, U_1 + r\sin\vartheta\, U_2 + zU_3 \\ &= r\,E_1 + z\,U_3; \end{aligned}$$

(ii) $$\begin{aligned} xU_1 + yU_2 + zU_3 &= \rho\cos\varphi\cos\vartheta\, U_1 + \rho\cos\varphi\sin\vartheta\, U_2 + \rho\sin\varphi\, U_3 \\ &= \rho F_1. \end{aligned}$$

If you cannot do this type of exercise by inspection, you may find the following method helpful. We can write the equations giving the F_i in terms of the U_i in matrix form as

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\vartheta & \cos\varphi\sin\vartheta & \sin\varphi \\ -\sin\vartheta & \cos\vartheta & 0 \\ -\sin\varphi\cos\vartheta & -\sin\varphi\sin\vartheta & \cos\varphi \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = M \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix};$$

Since the F_i and the U_i are both frames, M must be an orthogonal matrix, i.e. $M^{-1} = {}^tM$, so we can write $\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = {}^tM \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$, giving the U_i in terms of the F_i.

As an example, we work part (b)(ii):

$$\begin{aligned} \cos\vartheta\, U_1 + \sin\vartheta\, U_2 + U_3 &= (\cos\vartheta, \sin\vartheta, 1) \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \\ &= (\cos\vartheta, \sin\vartheta, 1)\,{}^tM \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \\ &= (\cos\vartheta, \sin\vartheta, 1) \begin{pmatrix} \cos\varphi\cos\vartheta & -\sin\vartheta & -\sin\varphi\cos\vartheta \\ \cos\varphi\sin\vartheta & \cos\vartheta & -\sin\varphi\sin\vartheta \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \\ &= (\cos\varphi + \sin\varphi, 0, \cos\varphi - \sin\varphi) \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \\ &= (\cos\varphi + \sin\varphi)F_1 + (\cos\varphi - \sin\varphi)F_3. \end{aligned}$$

II.7 CONNECTION FORMS

Introduction

These last two sections of Chapter II are of quite a different nature to the preceding ones. The material is more abstract, and concerned almost entirely with forms. We expect you will find these sections difficult. However, they are not needed again until Chapter VI, so you may be content with just a superficial understanding at this stage of the course. If necessary, you can revise these sections when you reach Chapter VI: you should find them easier going at that reading after the lapse of time.

Section II.7 draws together the previous two sections and Section I.5. Its purpose is to generalize the Frenet formulas to arbitrary frame fields. The matrix techniques introduced look messy but are simple to use in practice.

READ: Section II.7 (pages 85–90).

Comment

(i) *Page 88, Theorem 7.3* The ij-th entry of $dA\,{}^tA = \sum_k (dA)_{ik}({}^tA)_{kj} = \sum_k (da_{ik})(a_{jk})$, which we write as $\sum_k a_{jk}da_{ik}$ to keep to the standard convention of putting the differentials on the right. We cannot use such a conventional arrangement with the matrices dA and ${}^tA$, however, as matrix multiplication is not commutative.

Supplementary Comment

(i) *Page 89: dA* The fact that $d(\cos\vartheta) = -\sin\vartheta\, d\vartheta$, and the other similar results, follow from Lemma I.5.7.

Summary

Notation

ω_{ij}	Page 85, Lemma 7.1
ω	Page 87, line 4
A	Page 88, line 11
dA	Page 88, line 18

Definitions

(i)	Connection forms ω_{ij}	Page 85, Lemma 7.1
(ii)	Attitude matrix (A) of a frame field	Page 88, line 11
(iii)	Differential of a matrix whose entries are functions from E^3 to R	Page 88, line 18

Results

(i)	ω_{ij} is a 1-form and $\omega_{ij} = -\omega_{ji}$.	Page 85, Lemma 7.1
(ii)	The connection equations: $\nabla_V E_i = \Sigma\omega_{ij}(V)E_j$.	Page 86, Theorem 7.2
(iii)	$\omega = dA\,{}^tA$.	Page 88, Theorem 7.3

Technique

(i)	Calculation of the connection forms using $\omega = dA\,{}^tA$.	Page 88, Theorem 7.3

Exercises

Technique (i)

1. Page 91, Exercise 4, using A and ω.

Theory Exercise

2. Page 91, Exercise 5.

Solutions

1. Page 91, Exercise 4.

$$A = \begin{pmatrix} \cos\varphi\cos\vartheta & \cos\varphi\sin\vartheta & \sin\varphi \\ -\sin\vartheta & \cos\vartheta & 0 \\ -\sin\varphi\cos\vartheta & -\sin\varphi\sin\vartheta & \cos\varphi \end{pmatrix}$$

$$dA = \begin{pmatrix} -\sin\varphi\cos\vartheta\, d\varphi - \cos\varphi\sin\vartheta\, d\vartheta & -\sin\varphi\sin\vartheta\, d\varphi + \cos\varphi\cos\vartheta\, d\vartheta & \cos\varphi\, d\varphi \\ -\cos\vartheta\, d\vartheta & -\sin\vartheta\, d\vartheta & 0 \\ -\cos\varphi\cos\vartheta\, d\varphi + \sin\varphi\sin\vartheta\, d\vartheta & -\cos\varphi\sin\vartheta\, d\varphi - \sin\varphi\cos\vartheta\, d\vartheta & -\sin\varphi\, d\varphi \end{pmatrix}$$

$$\omega = dA^tA = \begin{pmatrix} 0 & \cos\varphi\, d\vartheta & d\varphi \\ -\cos\varphi\, d\vartheta & 0 & \sin\varphi\, d\vartheta \\ -d\varphi & -\sin\varphi\, d\vartheta & 0 \end{pmatrix}$$

so $\omega_{12} = \cos\varphi\, d\vartheta$

$\omega_{13} = d\varphi$

$\omega_{23} = \sin\varphi\, d\vartheta.$

2. Page 91, Exercise 5.

$$\nabla_V W = \nabla_V \big(\sum_i f_i E_i\big)$$

$$= \sum_i \nabla_V (f_i E_i), \quad \text{by Corollary II.5.4.(1),}$$

$$= \sum_i \big(V[f_i]\, E_i + f_i \nabla_V E_i\big), \quad \text{by Corollary II.5.4.(3),}$$

$$= \sum_i \big(V[f_i]\, E_i + f_i \sum_j \omega_{ij}(V) E_j\big), \quad \text{by the connection equations,}$$

$$= \sum_i V[f_j]\, E_j + \sum_j \sum_i f_i \omega_{ij}(V) E_j, \quad \text{by rearrangement,}$$

$$= \sum_j \big\{V[f_j] + \sum_i f_i \omega_{ij}(V)\big\} E_j.$$

In the case of the natural frame field all the connection forms are zero (prove!) and so this equation reduces to

$$\nabla_V W = \sum_j V[f_j]\, U_j$$

as given on page 79.

II.8 THE STRUCTURAL EQUATIONS

Introduction

As mentioned in the previous section, you will probably find this section rather abstract and hard going. Do not attempt it until you feel fairly confident about both Section II.6 and Section II.7. You should also be familiar with the ideas of *linear functional, dual space* and *dual basis* (see *Unit M201 12, Linear Functionals and Duality,* Sections 1 and 2).

This section deals with finding the exterior derivatives of the connection forms introduced in Section II.7 and of the *dual 1-forms* of a frame field. As we do not yet know what an exterior derivative is, this section, like I.6, must be approached purely formally, as manipulation of strings of symbols. In Chapter IV we shall find a meaning for the exterior derivative, and in Chapter VI we shall relate that to the results developed here.

The results of this section are not needed until Chapter VI. You will probably find it hard: if so, do not worry about understanding it in depth now, but return to it from time to time so that it becomes fairly familiar before Chapter VI is reached.

READ: Section II.8 (pages 91–95).

Comments

(i) *Page 91: dual 1-forms* In Lemma II.6.3 we saw that if E_1, E_2, E_3 is a frame field then any vector field V can be expressed as

$$V = (V \cdot E_1)E_1 + (V \cdot E_2)E_2 + (V \cdot E_3)E_3.$$

In a similar manner, we obtain the fact that, if v_p is a tangent vector,

$$v_p = (v_p \cdot E_1(p))E_1(p) + (v_p \cdot E_2(p))E_2(p) + (v_p \cdot E_3(p))E_3(p),$$

simply by applying Lemma II.1.5 to the frame $E_1(p)$, $E_2(p)$, $E_3(p)$. Thus if we define the functions θ_i from the set of all tangent vectors to E^3 to R (and extend to vector fields by the pointwise principle) by

$$\theta_i(v_p) = v_p \cdot E_i(p)$$

we may write

$$v_p = \theta_1(v_p)E_1(p) + \theta_2(v_p)E_2(p) + \theta_3(v_p)E_3(p)$$

$$V = \theta_1(V)E_1 + \theta_2(V)E_2 + \theta_3(V)E_3.$$

θ_i is certainly linear at each point, and so θ_i is a 1-form. In fact, since the E_i are orthonormal, we have

$$\theta_i(E_j) = \delta_{ij}$$

which is precisely what is meant by saying that the θ_i are the dual basis to the E_i. This is why we call the θ_i the dual 1-forms of the frame field.

(ii) *Page 94: Example* The θ_i have not been found yet, but a method for so doing is given on page 95. The ω_{ij} were computed in Exercise II.7.4.

Summary

Notation

θ_i	Page 91, Definition 8.1

Definition

(i)	Dual 1-form θ_i.	Page 91, Definition 8.1

Results

(i)	If V is a vector field, then $V = \Sigma\theta_i(V)E_i$.	Text, page 61
(ii)	$\theta_i(E_j) = \delta_{ij}$,	Text, page 61
(iii)	If ϕ is a 1-form, then $\phi = \Sigma\phi(E_i)\theta_i$.	Page 92, Lemma 8.2
(iv)	If $E_i = \sum_j a_{ij}U_j$, then $\theta_i = \sum_j a_{ij}\,dx_j$.	Page 92, line 20
(v)	The structural equations $d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$ $d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$.	Page 92, Theorem 8.3

Techniques

(i)	Finding the dual 1-forms θ_i by adhoc methods or using the attitude matrix.	Page 92, line 20
(ii)	Using the structural equations.	Page 92, Theorem 8.3

Exercises

Technique (i)

1. Page 96, Exercise 4. Part (c) depends on knowing what $\partial f/\partial r$, $\partial f/\partial\vartheta$ mean and on knowing the formula

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\vartheta}d\vartheta + \frac{\partial f}{\partial z}dz.$$

These will be explained in the final section of this text, so you may like to defer this part of the exercise until then.

Technique (ii)

2. Check the structural equations for the cylindrical frame field. (The connection forms were calculated on pages 89–90.)

Theory Exercise

3. Page 95, Exercise 1.

Solutions

1. Page 96, Exercise 4.

(a) $E_1 = \cos\vartheta\, U_1 + \sin\vartheta\, U_2$

$E_2 = -\sin\vartheta\, U_1 + \cos\vartheta\, U_2$

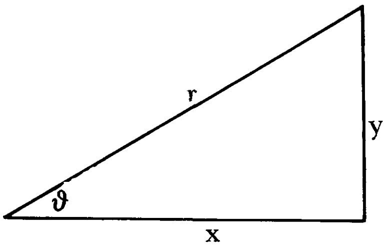
$E_3 = U_3$.

By definition, $\theta_1(U_j) = U_j \cdot E_1 = E_1 \cdot U_j$, so

$$\theta_1(U_1) = \cos\vartheta \qquad \theta_1(U_2) = \sin\vartheta \qquad \theta_1(U_3) = 0.$$

We compare this with the value of dr on each of the U_i.

$$dr(U_i) = U_i[r] = \frac{\partial r}{\partial x_i} \qquad \text{by Lemma I.3.2.}$$



$$r = (x^2 + y^2)^{\frac{1}{2}}$$

so

$$\frac{\partial r}{\partial x} = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{x}{r} = \cos\vartheta$$

$$\frac{\partial r}{\partial y} = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{y}{r} = \sin\vartheta$$

$$\frac{\partial r}{\partial z} = 0.$$

Thus

$$dr(U_1) = \cos\vartheta = \theta_1(U_1)$$

$$dr(U_2) = \sin\vartheta = \theta_2(U_2)$$

$$dr(U_3) = 0 \quad = \theta_3(U_3).$$

Since the 1-forms θ_1 and dr are equal on the frame field U_i they are equal on all tangent vectors and vector fields (by Lemma I.5.4), and so θ_1 = dr.

Similarly $\theta_2(U_j) = U_j \cdot E_2$, so

$$\theta_2(U_1) = -\sin\vartheta \qquad \theta_2(U_2) = \cos\vartheta \qquad \theta_2(U_3) = 0.$$

Also

$$d\vartheta(U_i) = U_i[\vartheta] = \frac{\partial\vartheta}{\partial x_i}.$$

$$\vartheta = \tan^{-1}\left(\frac{y}{x}\right)$$

so

$$\frac{\partial\vartheta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{-r\sin\vartheta}{r^2} = \frac{-\sin\vartheta}{r}$$

$$\frac{\partial\vartheta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\vartheta}{r^2} = \frac{\cos\vartheta}{r}$$

$$\frac{\partial\vartheta}{\partial z} = 0.$$

Thus

$$r\,d\vartheta(U_1) = -\frac{r\sin\vartheta}{r} = \theta_2(U_1)$$

$$r\,d\vartheta(U_2) = \frac{r\cos\vartheta}{r} = \theta_2(U_2)$$

$$r\,d\vartheta(U_3) = 0 = \theta_2(U_3)$$

and so $\theta_2 = r\,d\vartheta$.

$\theta_3(U_1) = 0 \qquad \theta_3(U_2) = 0 \qquad \theta_3(U_3) = 1$

$dz(U_1) = 0 \qquad dz(U_2) = 0 \qquad dz(U_3) = 1$

and so $\theta_3 = dz$.

(b) $E_i[r] = dr(E_i) = \theta_1(E_i) = \delta_{1i}$.

$$E_i[\vartheta] = d\vartheta(E_i) = \frac{\theta_2(E_i)}{r} = \frac{\delta_{2i}}{r}.$$

$E_i[z] = dz(E_i) = \theta_3(E_i) = \delta_{3i}$.

(c) $E_i[f] = df(E_i) = \left(\frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\vartheta}d\vartheta + \frac{\partial f}{\partial z}dz\right)(E_i)$ by the theorem to be proved in the final section,

$$= \frac{\partial f}{\partial r}\delta_{1i} + \frac{1}{r}\frac{df}{\partial\vartheta}\delta_{2i} + \frac{\partial f}{\partial z}\delta_{3i}.$$

Notice how we have used the Kronecker delta in (b) and (c) to convey several pieces of information in one statement. It does not matter if your solution did not use it, but bear in mind that you can often shorten the amount of writing by using it.

2. Now we check the structural equations.

$\theta_1 = dr$, $\theta_2 = r d\vartheta$, $\theta_3 = dz$,

$\omega_{12} = d\vartheta$, $\omega_{13} = 0$, $\omega_{23} = 0$ as found in Section II.7:

$d\theta_1 = d(dr) = 0$ and $\omega_{12}\wedge\theta_2 + \omega_{13}\wedge\theta_3 = d\vartheta\wedge r d\vartheta + 0 = 0$;

$d\theta_2 = d(r d\vartheta) = dr\wedge d\vartheta$ and $\omega_{21}\wedge\theta_1 + \omega_{23}\wedge\theta_3 = -d\vartheta\wedge dr + 0 = dr\wedge d\vartheta$;

$d\theta_3 = d(dz) = 0$ and $\omega_{31}\wedge\theta_1 + \omega_{32}\wedge\theta_2 = 0$;

$d\omega_{12} = d(d\vartheta) = 0$ and $\omega_{13}\wedge\omega_{32} = 0$;

$d\omega_{13} = 0$ and $\omega_{12}\wedge\omega_{23} = 0$;

$d\omega_{23} = 0$ and $\omega_{21}\wedge\omega_{13} = 0$.

3. Page 95, Exercise 1.

$$\begin{aligned}
d\phi &= d\Big(\sum_i f_i\theta_i\Big) \\
&= \sum_i d(f_i\theta_i) \\
&= \sum_i (df_i \wedge \theta_i + f_i d\theta_i), && \text{by Theorem I.6.4.(2),} \\
&= \sum_i \Big(df_i \wedge \theta_i + f_i \sum_j \omega_{ij} \wedge \theta_j\Big), && \text{by Theorem II.8.3.(1),} \\
&= \sum_j df_j \wedge \theta_j + \sum_j \sum_i f_i\omega_{ij} \wedge \theta_j, && \text{by rearrangement,} \\
&= \sum_j \Big\{df_j + \sum_i f_i\omega_{ij}\Big\} \wedge \theta_j.
\end{aligned}$$

PARTIAL DIFFERENTIATION

We have met two new coordinate systems in this chapter: r, ϑ, z and ρ, ϑ, φ. If you have met these before you may have come across expressions such as $\partial f/\partial r$, $\partial f/\partial \vartheta$ and equations such as

$$df = \frac{\partial f}{\partial \rho}\, d\rho + \frac{\partial f}{\partial \vartheta}\, d\vartheta + \frac{\partial f}{\partial \varphi}\, d\varphi.$$

Since the only partial derivatives we have so far formally defined are the $\partial f/\partial x_i$, where x_i are the natural coordinate functions, our purpose here is to give a meaning to expressions such as $\partial f/\partial r$, $\partial f/\partial \vartheta$ which is consistent with our idea of a partial derivative.

To do this properly, we must first of all be quite clear about the use of the symbols x, y, z, as defined on page 4. They denote not "dummy variables", but the natural coordinate function from E^3 to R. That is, if $p = (p_1, p_2, p_3)$ then $x(p) = p_1$, $y(p) = p_2$, $z(p) = p_3$. In particular $(x, y, z)(p) = (p_1, p_2, p_3) = p$, and so (x, y, z) is the identity function on E^3.

Now suppose that f is a function from E^3 to R. We must first notice that, although f is a *single* function from E^3 to R, it has different expressions in terms of x, y, z and in terms of ρ, ϑ, φ say. For example, let us consider the function $f = x^2 + y^2 + z^2$. We know $\rho^2 = x^2 + y^2 + z^2$, so we can also write $f = \rho^2$. Now, because (x, y, z) is the identity function, we can write

$$f = f \circ (x, y, z) = f(x, y, z)$$

but f is *not* equal to $f(\rho, \vartheta, \varphi)$ because

$$f(\rho, \vartheta, \varphi) = f \circ (\rho, \vartheta, \varphi) = \rho^2 + \vartheta^2 + \varphi^2.$$

Can we find a function g such that $f = g(\rho, \vartheta, \varphi)$, that is $f = g \circ m$ where m is the function $(\rho, \vartheta, \varphi)\colon E^3 \longrightarrow E^3$?

$$\begin{tikzcd} E^3 \arrow[rr, "f"] \arrow[dr, "m= (\rho,\vartheta,\varphi)"'] & & R \\ & E^3 \arrow[ur, "g"'] & \end{tikzcd}$$

In this case we can do so quite simply. If $p \in E^3$ then

$$f(p) = \rho^2(p) = (\rho(p))^2$$

$$g(\rho, \vartheta, \varphi)(p) = g(\rho(p), \vartheta(p), \varphi(p))$$

and so we require the function g to pick out the first coordinate and square it: but the function that does this is precisely the function we have called x^2. Thus $g = x^2$ and so

$$f = x^2 \circ m.$$

We return to the general case, and recall that the definition of $\partial f/\partial x$ is

$$\frac{\partial f}{\partial x}(p) = \frac{d}{dt} f(t, p_2, p_3)\bigg|_{t=p_1} = \frac{d}{dt} f(t, y(p), z(p))\bigg|_{t=x(p)}.$$

Roughly speaking, we are holding y and z fixed and differentiating with respect to x. If we want to define $\partial f/\partial\rho$ in a similar fashion we need to hold ϑ and φ fixed and differentiate with respect to ρ. This is where we need to use the function g: since

$$f = g(\rho, \vartheta, \varphi)$$

we define $\dfrac{\partial f}{\partial\rho}$ by

$$\frac{\partial f}{\partial\rho}(p) = \frac{d}{dt}g(t, \vartheta(p), \varphi(p))\Big|_{t=\rho(p)}.$$

But this is precisely $\partial g/\partial x$ evaluated at $(\rho(p), \vartheta(p), \varphi(p))$, because, roughly speaking, $\partial g/\partial x$ means "differentiate with respect to the first variable" no matter what the "variables" may be called.

This is formalized in the following definition.

Definition

If $m = (m_1, m_2, m_3)$ is a differentiable one-to-one function from a subset of E^3 to E^3 with differentiable inverse m^{-1}, and f is a differentiable function from E^3 to R, and g is a function such that $f = g \circ m$ (in fact g must be $f \circ m^{-1}$), then

$$\frac{\partial f}{\partial m_i}(p) = \frac{\partial g}{\partial x_i}(m(p)).$$

Notice that this definition of $\partial f/\partial m_i$ is dependent on the whole function m: if $\bar{m} = (m_1, \bar{m}_2, \bar{m}_3)$ then $\partial f/\partial m_1$ may be different depending on whether m or $\bar{m}$ is being used.

In practice this definition is very easy to work with. In our previous example we had $f = \rho^2$, $g = x^2$, and so $\partial f/\partial\rho = (\partial g/\partial x)(m) = 2x(\rho, \vartheta, \varphi) = 2\rho$. Another example: if $f = \rho\cos\varphi$ then $g = x\cos z$, and

$$\frac{\partial f}{\partial\rho} = \frac{\partial g}{\partial x}(m) = \cos z(\rho, \vartheta, \varphi) = \cos\varphi;$$

$$\frac{\partial f}{\partial\vartheta} = \frac{\partial g}{\partial y}(m) = 0;$$

$$\frac{\partial f}{\partial\varphi} = \frac{\partial g}{\partial z}(m) = -x\sin z\,(\rho, \vartheta, \varphi) = -\rho\sin\varphi.$$

This should show you that this process effectively treats ρ, ϑ, φ as "dummy variables" just as x, y, z are normally treated, and so the new partial derivatives can be written down without the step of introducing g.

For the usual partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ we had the result (Corollary I.5.5):

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

Now that we have defined other partial derivatives suitably, we can prove an analogous theorem.

Theorem

If $m = (m_1, m_2, m_3)$ is a differentiable one-to-one function from a subset of E^3 to E^3 with differentiable inverse m^{-1}, and f is a differentiable function from E^3 to R, then

$$df = \Sigma \frac{\partial f}{\partial m_i} dm_i.$$

In particular,

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \vartheta} d\vartheta + \frac{\partial f}{\partial z} dz;$$

$$df = \frac{\partial f}{\partial \rho} d\rho + \frac{\partial f}{\partial \vartheta} d\vartheta + \frac{\partial f}{\partial \varphi} d\varphi.$$

(Corollary I.5.5 is the particular case of this theorem for $m_i = x_i$.)

Proof

We use the chain rule given in the additional text on Section I.7. We can take $g = f \circ m^{-1}$ to obtain

$$f = g \circ m.$$

Now the chain rule gives

$$f_* = g_*(m)m_*.$$

If v is a tangent vector at the point p, this gives

$$f_*(v) = g_*(m(p))m_*(v), \qquad (*)$$

where g_* is written as a matrix evaluated at m(p) and the vectors $f_*(v)$, $m_*(v)$ are written as column vectors. Now, Theorem I.7.5 tells us that

$$f_*(v) = (v[f]) = (df(v)):$$

this is a 1 × 1 matrix so that corresponding column vector is also (df(v)). The same theorem tells us that

$$\begin{aligned} m_*(v) &= (v[m_1], v[m_2], v[m_3]) \\ &= (dm_1(v), dm_2(v), dm_3(v)), \end{aligned}$$

which is

$$\begin{pmatrix} dm_1(v) \\ dm_2(v) \\ dm_3(v) \end{pmatrix}$$

as a column vector.

By Corollary I.7.7 g_* has matrix

$$\left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}\right).$$

Evaluating this at the point m(p) we have

$$\left(\frac{\partial g}{\partial x_1}(m(p)), \frac{\partial g}{\partial x_2}(m(p)), \frac{\partial g}{\partial x_3}(m(p))\right)$$

which is precisely

$$\left(\frac{\partial f}{\partial m_1}(p), \frac{\partial f}{\partial m_2}(p), \frac{\partial f}{\partial m_3}(p)\right)$$

by the definition of $(\partial f/\partial m_i)$. Putting the expressions we have obtained into (*):

$$\begin{aligned}(df(v)) &= \left(\frac{\partial f}{\partial m_1}(p), \frac{\partial f}{\partial m_2}(p), \frac{\partial f}{\partial m_3}(p)\right)\begin{pmatrix} dm_1(v) \\ dm_2(v) \\ dm_3(v)\end{pmatrix} \\ &= \left(\frac{\partial f}{\partial m_1}(p)dm_1(v) + \frac{\partial f}{\partial m_2}(p)dm_2(v) + \frac{\partial f}{\partial m_3}(p)dm_3(v)\right) \\ &= \left(\left(\frac{\partial f}{\partial m_1}dm_1 + \frac{\partial f}{\partial m_2}dm_2 + \frac{\partial f}{\partial m_3}dm_3\right)(v)\right).\end{aligned}$$

This is true for all tangent vectors **v**, and so

$$df = \frac{\partial f}{\partial m_1}dm_1 + \frac{\partial f}{\partial m_2}dm_2 + \frac{\partial f}{\partial m_3}dm_3.$$

You do not need to know any details of this proof. All you need for this course is the defintion of general partial derivatives and the result stated in the theorem.

Summary

Notation

$\left\{\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \vartheta}, \frac{\partial f}{\partial z}\right\}$	Text, page 66
$\left\{\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \vartheta}, \frac{\partial f}{\partial \varphi}\right\}$	Text, page 66

Definition

(i)	Partial derivatives	Text, page 67

Results

(i)	$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \vartheta}d\vartheta + \frac{\partial f}{\partial z}dz.$	Text, page 68
(ii)	$df = \frac{\partial f}{\partial \rho}d\rho + \frac{\partial f}{\partial \vartheta}d\vartheta + \frac{\partial f}{\partial \varphi}d\varphi.$	Text, page 68

Techniques

(i)	Finding partial derivatives.	Text, page 67
(ii)	Using the expressions for df in terms of the partial derivatives.	Text, page 68

Exercises

Technique (i)

1. Find $\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \vartheta}, \frac{\partial f}{\partial \varphi}$ when $f = \rho \cos\vartheta \cos\varphi$.

2. Find $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \vartheta}, \frac{\partial f}{\partial z}$ when $f = x^2 + z^2$.

Technique (ii)

This has already been tested in Exercise II.8.4(c). Do that exercise now if you previously deferred it.

Solutions

1. $f = \rho\cos\vartheta \cos\varphi$

$$= x \cos y \cos z \ (\rho, \vartheta, \varphi)$$

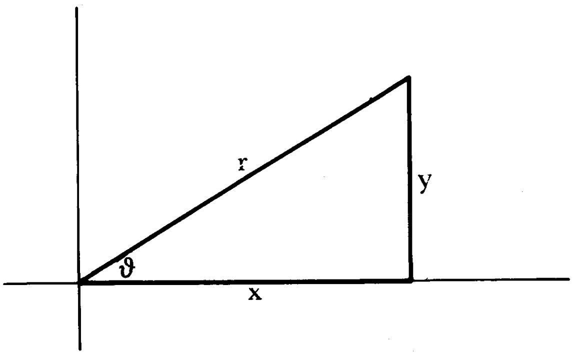
$$= g(\rho, \vartheta, \varphi) \qquad \text{where } g = x \cos y \cos z.$$

$$\frac{\partial f}{\partial \rho} = \frac{\partial g}{\partial x}(\rho, \vartheta, \varphi) = \cos y \cos z \ (\rho, \vartheta, \varphi) = \cos\vartheta \cos\varphi.$$

$$\frac{\partial f}{\partial \vartheta} = \frac{\partial g}{\partial y}(\rho, \vartheta, \varphi) = -x \sin y \cos z \ (\rho, \vartheta, \varphi) = -\rho \sin\vartheta \cos\varphi.$$

$$\frac{\partial f}{\partial \varphi} = \frac{\partial g}{\partial z}(\rho, \vartheta, \varphi) = -x \cos y \sin z \ (\rho, \vartheta, \varphi) = -\rho \cos\vartheta \sin\varphi.$$

2. We must first of all express f in terms of r, ϑ, z.



From the diagram we see that

$$x = r \cos\vartheta$$

and so

$$f = x^2 + z^2$$
$$= r^2 \cos^2 \vartheta + z^2.$$

As before, we can introduce $g = x^2 \cos^2 y + z^2$, or we can go directly to the results:

$$\frac{\partial f}{\partial r} = 2r \cos^2 \vartheta,$$

$$\frac{\partial f}{\partial \vartheta} = -2r^2 \cos \vartheta \ \sin \vartheta;$$

$$\frac{\partial f}{\partial z} = 2z.$$

FURTHER EXERCISES AND SOLUTIONS

Section II.1

Recommended Exercises

Page 49, Exercise 2. This shows that Euclidean distance is a metric in the sense of M202 and M231.

Page 50, Exercise 11. This relates the vector operation grad to the norm and dot product.

Solutions

2(a). $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = (\Sigma(p_i - q_i)^2)^{\frac{1}{2}}$, which is non-negative.

$d(\mathbf{p}, \mathbf{q}) = 0 \Rightarrow \Sigma(p_i - q_i)^2 = 0$, and a sum of squares is zero if and only if each term is zero, that is $p_i = q_i$.

2(b). $(p_i - q_i)^2 = (q_i - p_i)^2$ so $d(\mathbf{p}, \mathbf{q}) = d(\mathbf{q}, \mathbf{p})$.

2(c). $d(\mathbf{p}, \mathbf{q}) + d(\mathbf{q}, \mathbf{r}) = \|\mathbf{p} - \mathbf{q}\| + \|\mathbf{q} - \mathbf{r}\| \geqslant \|\mathbf{p} - \mathbf{q} + \mathbf{q} - \mathbf{r}\| = \|\mathbf{p} - \mathbf{r}\| = d(\mathbf{p}, \mathbf{r})$.

11(a). $df(\mathbf{v}) = \Sigma\, v_i \dfrac{\partial f_i}{\partial x_i} = (\Sigma v_i U_i(\mathbf{p}))\cdot(\Sigma \dfrac{\partial f_i}{\partial x_i} U_i(\mathbf{p})) = \mathbf{v}\cdot\nabla f(\mathbf{p})$.

11(b). Put $\mathbf{w} = (\nabla f)(\mathbf{p})$.
$\mathbf{u}[f] \quad = \mathbf{u}\cdot\nabla f(\mathbf{p}) = \mathbf{u}\cdot\mathbf{w}$

and $|\mathbf{u}\cdot\mathbf{w}| \leqslant \|\mathbf{u}\|\,\|\mathbf{w}\| = \|\mathbf{w}\|$

with equality when $\mathbf{u} = \dfrac{\mathbf{w}}{\|\mathbf{w}\|}$.

Section II.2

Technique Exercise

Page 55, Exercise 1.

Other Recommended Exercises

Page 55, Exercise 5. This deal with unit-speed reparametrizations in the sense of Theorem II.2.1.

Page 56, Exercise 10. This shows that arc-length is preserved by reparametrizations which are symmetric in the sense of Comment (i).

Solutions

1(a). $(2, 2t, t^2)_{\alpha(t)}$, $(2, 2, 1)_{(2, 1, \frac{1}{3})}$;

$t^2 + 2, 3$;

$(0, 2, 2t)_{\alpha(t)}$, $(0, 2, 2)_{(2, 1, \frac{1}{3})}$.

1(b). $s(t) = 2t + \frac{t^3}{3}$; $\frac{14}{3}$.

5. If β_i is the unit-speed reparametrization based at $t = t_i$, and s_i is the arc-length function based at $t = t_i$,

$$\alpha(t) = \beta_1(s_1(t)) = \beta_2(s_2(t)).$$

$$\begin{aligned} s_1(t) &= \int_{t_1}^{t} v(u)du \\ &= \int_{t_1}^{t_2} v(u)du + \int_{t_2}^{t} v(u)du \\ &= s_0 + s_2(t) \end{aligned}$$

where s_0 is the arc-length of α from t_1 to t_2.

10. Put $\beta = \alpha(h)$.

(a) Arc-length of β from a to b

$$\begin{aligned} &= \int_a^b v_\beta(u)du \\ &= \int_{h^{-1}(c)}^{h^{-1}(d)} h'(u)v_\alpha(h(u))du \\ &= \int_c^d v_\alpha(t)dt \end{aligned}$$

= arc-length of α from c to d.

(b) Arc-length of β from a to b

$$\begin{aligned} &= \int_a^b v_\beta(u)du \\ &= \int_{h^{-1}(d)}^{h^{-1}(c)} (-h'(u))v_\alpha(h(u))du \\ &= -\int_d^c v_\alpha(t)dt \\ &= \int_c^d v_\alpha(t)dt. \end{aligned}$$

Section II.3

Technique Exercises

Page 63–64, Exercises 2, 3.

Other Recommended Exercises

Page 64, Exercise 5. This expresses all the Frenet formulas in a single format.

Page 64, Exercise 6. This deals with the approximation of a curve by part of a circle.

Solutions

2. $$T(s) = \left(\frac{(1+s)^{\frac{1}{2}}}{2}, \frac{-(1-s)^{\frac{1}{2}}}{2}, \frac{1}{\sqrt{2}}\right)$$

$$N(s) = \left(\frac{(1-s)^{\frac{1}{2}}}{\sqrt{2}}, \frac{(1+s)^{\frac{1}{2}}}{\sqrt{2}}, 0\right)$$

$$B(s) = \left(\frac{-(1+s)^{\frac{1}{2}}}{2}, \frac{(1-s)^{\frac{1}{2}}}{2}, \frac{1}{\sqrt{2}}\right)$$

$$\kappa(s) = \tau(s) = \frac{1}{2\sqrt{2}(1-s^2)^{\frac{1}{2}}}$$

3. Evaluate both sides of each equation.

5. $A \times T = \kappa B \times T = \kappa N$

$A \times N = \tau T \times N + \kappa B \times N = \tau B - \kappa T$

$A \times B = \tau T \times B = -\tau N.$

6. The only solution for γ is

$$\gamma(s) = c - r \cos\frac{s}{r} N_0 + r \sin\frac{s}{r} T_0$$

where $c = \dfrac{N_0}{\kappa_0} + \beta(0)$

$$r = \frac{1}{\kappa_0}.$$

Section II.4

Technique Exercises

Pages 74–75, Exercises 1(a), 3(a), 4 (first part), 6, 7, 9(a), (b).

Other Recommended Exercises

Page 74, Exercise 5. This gives a formula for κ.

Page 75, Exercise 13. This introduces another special curve derived from a given curve. Since one of a, $\bar{a}$ is negative, you will first have to adapt Example II.3.3 for the case where a is negative.

Solutions

1(a). $T(t) = \frac{1}{2+t^2}(2, 2t, t^2)$

$N(t) = \frac{1}{2+t^2}(-2t, 2-t^2, 2t)$

$B(t) = \frac{1}{2+t^2}(t^2, -2t, 2)$

$\kappa(t) = \tau(t) = \frac{2}{(2+t^2)^2}$.

3(a). $T(0) = \frac{(1, 0, 1)}{\sqrt{2}}$

$N(0) = (0, 1, 0)$

$B(0) = \frac{(-1, 0, 1)}{\sqrt{2}}$

$\kappa(0) = 1$

$\tau(0) = \frac{3}{4}$.

4. Evaluate both sides of each equation.

6. $\alpha' = cT$.

$\alpha'' = cT' = c^2\kappa N$.

$\alpha' \times \alpha'' = c^3\kappa B$.

Now substitute in Theorem II.4.3.

7. $T(t) = \left(-\frac{a}{c}\sin t, \frac{a}{c}\cos t, \frac{b}{c}\right)$

$N(t) = (-\cos t, -\sin t, 0)$

$B(t) = \left(\frac{b}{c}\sin t, -\frac{b}{c}\cos t, \frac{a}{c}\right)$

$\kappa(t) = \frac{a}{c^2}$

$\tau(t) = \frac{b}{c^2}$.

9(a). $\mathbf{u} = \frac{(1, 0, 1)}{\sqrt{2}}$

$\vartheta = \frac{\pi}{4}$

$\gamma(t) = \left(t - \frac{t^3}{6}, t^2, -t + \frac{t^3}{6}\right)$.

9(b). $\mathbf{u} = (0, 0, 1)$

$\vartheta = \frac{\pi}{4}$

$\gamma(t) = (3t - t^3, 3t^2, 0)$.

5. $$\alpha'' = \frac{dv}{dt} T + \kappa v^2 N$$

$$\|\alpha''\|^2 = \left(\frac{dv}{dt}\right)^2 + (\kappa v^2)^2.$$

13. If $a > 0$, then $\kappa(s) = \frac{a}{c^2}$, $N(s) = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0)$;

if $a < 0$, then $\kappa(s) = -\frac{a}{c^2}$, $N(s) = (\cos\frac{s}{c}, \sin\frac{s}{c}, 0)$.

Substitute these to obtain

$$\beta_{ab}{}^* = \beta_{\bar{a}b}$$

for all a. Now $\overline{(\bar{a})} = a$ so

$$\beta_{\bar{a}b}{}^* = \beta_{ab}.$$

Section II.5

Technique Exercises

Page 80, Exercises 1(b), 2(b), (d), (e), (f), 4.

Other Recommended Exercises

Page 80, Exercise 3, This extends the result that if Y is a vector field on a curve and $\|Y\|$ is constant then $Y \cdot Y' = 0$.

Page 80, Exercise 6(b). This deals with the definition of covariant derivative.

Page 80, Exercise 7. This exercise introduces the bracket of two vector fields.

Solutions

1(b). $(1, 2, 4)_p$

2(b). $-y\,U_3$.

2(d). $-\sin x\ U_1 + \cos x\ U_3$.

2(e). $-y^2 \cos x\ U_1 - y^2 \sin x\ U_2$.

2(f). $(y^2 - x\cos x - yz\sin x)U_1 + (-x\sin x + yz\cos x)U_2 - 2xy\,U_3$.

4. $X = \Sigma x_i U_i$.

$$\nabla_V X = \sum_i V[x_i]\,U_i = \sum_i \sum_j v_j \frac{\partial x_i}{\partial x_j} U_i = \sum_i \sum_j v_j\,\delta_{ij}\,U_i$$

$$= \sum_i v_i U_i = V.$$

3. $W(p+tv) \cdot W(p+tv) = \text{constant}$

$\Rightarrow W(p+tv) \cdot W(p+tv)' = 0$

$\Rightarrow W(p) \cdot \nabla_v W = 0.$

6(b). $W(\alpha)'(t) = \nabla_{\alpha'(t)} W = \nabla_v W$.

7(a). Both sides equal $\sum_i \sum_j \left(v_j \frac{\partial w_i}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j} \right) \frac{\partial f}{\partial x_i}$.

7(b). The definition is antisymmetric.

7(c). $[U,[V,W]][f] = U[V,W][f] - [V,W]U[f]$

$= UVW[f] - UWV[f] - VWU[f] + WVU[f]$.

Adding the other two similar expressions to this gives zero.

7(d). $[fV,gW][h] = fV[gW[h]] - gW[fV[h]]$

$= f(V[g]W[h] + gVW[h]) - g(W[f]V[h] + fWV[h])$

$= fV[g]W[h] - gW[f]V[h] + fg[V,W][h]$.

Section II.6

Techinque Exercises

Pages 84–85, Exercises 3, 4.

Solutions

3. Choose any unit E_2 orthogonal to E_1 and put $E_3 = E_1 \times E_2$. For example

$E_2 = \sin z\ U_2 - \cos z\ U_3$

$E_3 = -\sin x\ U_1 + \cos x \cos z\ U_2 + \cos x \sin z\ U_3$.

4. The same formulas are obtained as for the spherical frame field.

Section II.7

Technique Exercises

Pages 90–91, Exercises 1, 2, 3, 7.

Solutions

1. $\omega_{12} = 0; \omega_{13} = \omega_{23} = \frac{df}{\sqrt{2}}$.

2. All zero.

3. A is orthogonal.

$\omega_{12} = -df$;

$\omega_{13} = \cos f\ df$;

$\omega_{23} = \sin f\ df$.

7(a). For any f,

$$F_1[f] = \cos\varphi\ \cos\vartheta\ \frac{\partial f}{\partial x} + \cos\varphi\ \sin\vartheta\ \frac{\partial f}{\partial y} + \sin\varphi\ \frac{\partial f}{\partial z}.$$

$$\rho = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$\vartheta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\varphi = \tan^{-1}\left(\frac{z}{(x^2+y^2)^{\frac{1}{2}}}\right):$$

evaluate the partial derivatives and substitute.

7(b). $-\sin\rho\ F_2 + \cos\rho\ F_3$.

Section II.8

No further exercises are recommended.

M334 II

DIFFERENTIAL GEOMETRY

I Calculus on Euclidean Space

II Frame Fields

III Euclidean Geometry

IV Calculus on a Surface

V Shape Operators

VI Geometry of Surfaces in E^3